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# Analytical Treatment of Two-Dimensional Supersonic Flow. I. Shock-Free Flow

J. J. Mahony and R. E. Meyer

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# ANALYTICAL TREATMENT OF TWO-DIMENSIONAL SUPERSONIC FLOW

## I. SHOCK-FREE FLOW

By J. J. MAHONY* and R. E. MEYER†

*Department of Mathematics, University of Manchester*

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CONTENTS



A solution is presented for the general, wave-interaction problem of steady, irrotational, homentropic flow of a perfect gas. It can be interpreted as a convergent process of successive approximations, based on the solution of linearized theory, for shock-free flow. It constitutes an approximate solution for flow with weak shocks.

In part I, the equations of motion are transformed into linear equations in characteristic variables. Their solution is of different type according to whether the flow is near-sonic, hypersonic, or in between these extremes. Special attention is given to the types of boundary condition which occur in physical problems, and solution methods are devised to cope with these types in the medium Mach number range. The method of Riemann functions is used to calculate accurately the pressure distribution in the first interaction region of a jet expanding from a perfect nozzle. It is shown by the help of double power series that shock waves will always occur in the first 'period' of such a jet, even for pressure ratios arbitrarily near unity. The Riemann function approach is also shown to be suitable for the approximate calculation of the flow past aerofoils of prescribed shape; when the requirements of accuracy are exacting, the method of double power series expansion presents the problem in a form suited to high-speed digital computers.

## 1. Introduction

A solution for the problem of plane waves of finite amplitude was given by Riemann (1860), but the attempts to find a similar, analytical solution for the closely analogous problem of steady, two-dimensional, supersonic, homentropic and irrotational flow of a perfect gas have not been successful. An analytical solution for the special case of the simple wave was given by Meyer (1908), and a purely numerical method of solution for the general problem

* Now at the Aeronautical Research Laboratories, Melbourne.

† Now at the William & Agnes Bennett Supersonic Laboratory, Aeronautics Department, University of Sydney.

was formulated by Massau (1900) and Busemann (1929). The analytical hodograph method was pioneered by Chaplygin (1904) and developed by Lighthill (1947) and Cherry (1947, 1953), but great difficulties are still encountered in the adaptation of the solution to the boundary conditions of physical problems. The solution by this method of the basic, supersonic problem of the interaction of two simple waves has not yet been given.

It has been necessary, therefore, to rely largely on the approximate linearized theory introduced by Ackeret (1925). It is versatile as far as the adaptation to physical boundary conditions is concerned, but the approximation is not sufficient to account for important 'non-linear' features of the solutions, such as shock formation. Accordingly, attempts have been made to extend this theory to higher approximations. Such extensions are particularly important for the more general problems of steady, supersonic, axially symmetrical flow, spherical waves of finite amplitude, and similar problems, where the numerical method of characteristics is cumbersome and to which the hodograph method cannot be extended. For these problems the extension has met with particular difficulties, and it has indeed been shown (Meyer 1948) that the solution of linearized theory for the axially symmetrical flow is not the first of a convergent sequence of successive approximations. Some of the shortcomings of linearized theory have been removed in the 'uniform' first-order theories of Whitham (1952) and Meyer (1952), but the extension of these theories to higher-order approximations has not yet been achieved. A notable exception is the theory of Chen (1953) for the neighbourhood of the axis in axially symmetrical, supersonic flow. It is an important feature of the present method of solution that it can be generalized to steady, supersonic, axially symmetrical flow, to spherical waves of finite amplitude, etc. In fact, the present investigation has been conceived as a pilot experiment for the treatment of these more complicated problems.

## 1·1.

The general solution of the equations of motion is here obtained in two stages. In §2 the equations are reduced to a pair of new equations by a transformation suggested by an investigation of the structure of the solutions (Meyer 1949). The aim of the transformation is twofold. First, the equations are brought into a form the analytical solution of which is not complicated by the singularities of the correspondence between the flow plane and the characteristic plane (or the hodograph plane). Secondly, certain analytic elements which the earlier investigation (Meyer 1949) has shown to be common to all solutions are absorbed into the transformation, so that the new equations may be regarded as formulating a reduced problem which is less sensitive to the introduction of approximations than the original problem.

The reduced equations ((14) and (15), or (17) and (18)) are linear hyperbolic differential equations of standard type, and a number of solution methods are available of which Riemann's method and the expansion in double power series with respect to the characteristic variables are found to be well suited to physical boundary conditions. The coefficients of the equations are regular except for poles at the sonic speed and at the vacuum speed, and the solutions therefore differ somewhat in the cases of near-sonic flow, hypersonic flow and flow in the Mach number range between these extremes. A first approximation to the Riemann functions is given for the extreme cases (§§2·4, 2·5), but for the remainder of the paper, attention is confined to the medium Mach number range.* The Riemann functions are found by a modification of Picard's iteration process, which converges rapidly for problems involving relatively small velocity perturbations, and a high approximation to them is given explicitly (§ 2·1·1).

Branch lines (Lighthill 1947) may occur, at which the solution is singular; these singularities can be removed by a change of independent variables, but it may often be more convenient to proceed directly by the help of a particular solution satisfying the singular part of the boundary conditions. This solution is given in § 2·3 in the form of power series in one characteristic variable with coefficients depending on the other.

The solution is not singular at limit lines, but indicates them clearly when they occur.

## 1·2.

In developing the present method of solution, much attention has been given to the boundary conditions most commonly encountered in physical problems. In the theory of two-dimensional, inviscid fluid flow, whether subsonic or supersonic, the boundary conditions may be divided into three classes according to whether they are prescribed on a boundary the position of which is (i) known in both the hodograph plane and the flow plane, or (ii) known only in the hodograph plane, or (iii) unknown in the hodograph plane.

For supersonic flow, examples of the first class are the Cauchy boundaries and the characteristic boundaries. For problems involving only these, the solution of the reduced equations can be obtained directly by quadrature (Courant & Hilbert 1937), once the Riemann functions are known. As an example, the first wave interaction in the supersonic jet expanding from a perfect nozzle is treated in §§ 3 and 3·0·1. A high degree of accuracy is achieved with relatively little labour.

An example of the second class of boundary conditions is a free streamline (jet boundary), on which the pressure is a prescribed constant, but the stream direction is not known *a priori*, so that the boundary is 'firm' in the hodograph plane, but 'floating' in the flow plane. For this class of boundary conditions, Riemann's method does not furnish the solution directly, but leads to an integral equation, in the first place, and the method of double power series will often be preferable, particularly when an approximation of high order is required. The first 'period' of the supersonic jet, which contains a reflexion of a simple wave from the jet boundary, is treated in §§ 3·1 to 3·1·2. At first sight, the method of double power series appears little suited to this problem, which involves three wave interactions requiring each a different pair of such series. In fact, the method is found convenient and powerful. The first term of the series yields a solution equivalent to Hasimoto's (1953) and superior to that of linearized theory (Prandtl 1904, Pack 1950). Two terms suffice to explain shock formation and aperiodicity, except in a small range of Mach numbers, where more terms need be considered.

The curved streamline of prescribed shape in the flow plane, but on which the pressure is not given, affords an example of a boundary condition of the third class, and this is the most important type of boundary condition arising from physical and engineering problems. It corresponds to the 'direct problem' of aerofoil theory. Here, Riemann's method leads

* For its extent, see end of § 2·1·1.

always to an integral equation for the pressure distribution on the streamline. Additional difficulties, however, are encountered with this class of boundary conditions in both subsonic and supersonic flow. The differential equations are linear only when the velocity components or the characteristic variables are employed as independent variables.* The boundary conditions, on the other hand, are most naturally formulated in terms of the position co-ordinates in the flow plane as independent variables. If an analytical solution method is to be generally useful, a way needs to be found of reconciling these conflicting requirements. As an example, the reflexion of a simple wave from a curved streamline is treated in §§ 3·2 to 3·2·3. The conflict of requirements is seen to lead to a danger of the results being marred by lack of detail, when approximations of only a relatively low order are desired, and it is shown how the difficulty can be resolved by application of an apparently inconsistent procedure. To clarify the situation, the 'thin-body' concept is applied with precision (§ 3·2·1), and it is found to imply small perturbation velocity everywhere, but not small velocity gradients (§ 3·2·3), in contrast to the assumptions underlying linearized theory. The present theory also furnishes the extension to the general wave-interaction problem of Whitham's (1952) uniform first-order theory (§ 3·2·3).

The Riemann method is best suited to problems involving floating boundaries in the hodograph plane, when approximations of a low order are desired; but when a very high accuracy is required, the method of double power series is preferable. Its application to this type of problem is briefly discussed in § 3·3. The method is found to transform the differential equations and boundary conditions into a sequence of systems of linear, algebraic equations which can be solved successively. This form of the problem may be better suited to high-speed computing machines than forms modelled on the numerical method of characteristics (Clippinger & Gerber 1950; Courant, Isaacson & Rees 1952).

## 2. Basic equations

The equations of motion for the two-dimensional, steady, irrotational, homentropic, supersonic flow of a perfect gas can be written in the following, 'characteristic', form (Howarth 1953):

$$\mathrm{d}y/\mathrm{d}x = \tan(\theta-\mu), \quad \theta+t=\alpha=\text{const.}, \quad \text{on any 'plus' Mach line,} \tag{1}$$

$$\mathrm{d}y/\mathrm{d}x = \tan(\theta+\mu), \quad \theta-t=\beta=\text{const.}, \quad \text{on any 'minus' Mach line,} \tag{2}$$

where $x$, $y$ are Cartesian position co-ordinates in the plane of the flow; $q$ and $\theta$ are the polar components of velocity; $\mu$ is the local Mach angle,

$$t=\int_{a_s}^{q}\frac{\cot\mu}{q}\,\mathrm{d}q=\mu-\lambda\,\mathrm{arc\ tan}\,(\lambda\tan\mu)+\tfrac{1}{2}\pi(\lambda-1), \tag{3}$$

with $\lambda^2=(\gamma+1)/(\gamma-1)$; $\gamma$ is the ratio of the specific heats; and $a_s$ is the critical speed of sound. The energy equation,

$$q^2\left(\frac{1}{2}+\frac{\sin^2\mu}{\gamma-1}\right)=\frac{\gamma+1}{2(\gamma-1)}\,a_s^2=\text{const.}, \tag{4}$$

relates $q$ and $\mu$, so that $t$, $q$, $\mu$ are functions of one another.

* Even in problems like that of spherical waves of finite amplitude, where the differential equations are non-linear in any case, the hyperbolic character of the equations demands characteristic independent variables for their natural formulation.

The fundamental role played by the Mach lines suggests employing the 'characteristic' variables, $\alpha$ and $\beta$, as independent variables. The left-hand set of equations (1) and (2) can then be written in the more explicit form

$$\partial x/\partial\beta = h_\beta \cos(\theta-\mu), \quad \partial y/\partial\beta = h_\beta \sin(\theta-\mu), \tag{5}$$

$$\partial x/\partial\alpha = h_\alpha \cos(\theta+\mu), \quad \partial y/\partial\alpha = h_\alpha \sin(\theta+\mu), \tag{6}$$

where $h_\alpha(\alpha,\beta)$ and $h_\beta(\alpha,\beta)$ are parameters defined by these equations themselves. The transformation to characteristic independent variables is analogous to the hodograph transformation and, like that transformation, leads to a set of linear equations for $x$ and $y$.

If the positive direction on a Mach line is defined as that making an acute angle with the stream direction, then equations (5) and (6) imply that $h_\beta \mathrm{d}\beta$ and $h_\alpha \mathrm{d}\alpha$ are the elements of length, in the physical plane, in the positive directions of the plus and minus Mach lines, respectively. A physical interpretation of these characteristic length parameters is obtained by noting that the components in the positive Mach directions of the gradient of polar velocity components are

$$\partial q/h_\beta\partial\beta = -q\tan\mu/(2h_\beta), \quad \partial q/h_\alpha\partial\alpha = q\tan\mu/(2h_\alpha), \tag{7}$$

$$\partial\theta/h_\beta\partial\beta = 1/(2h_\beta), \quad \partial\theta/h_\alpha\partial\alpha = 1/(2h_\alpha). \tag{8}$$

That is, the parameters are inversely proportional to the components of velocity gradient in the positive Mach directions.

The length parameters $h_\alpha$ and $h_\beta$ are particularly closely related to the mathematical structure of the flow (Meyer 1949), and the components of velocity gradient* in the Mach directions are particularly closely related to the physical structure of the flow (Meyer 1952). This suggests employing as dependent variables $h_\alpha$ and $h_\beta$, rather than $x$ and $y$. A mathematical advantage of this choice of variables is that singularities of the correspondence between the flow plane and the characteristic plane can be handled with ease. They have been discussed by Craggs (1948), Meyer (1949), and Stocker & Meyer (1951). Limit lines appear as lines $h_\alpha = 0$ or $h_\beta = 0$ and need no attention, within the framework of equations (9) and (10); for shock waves see part II. Branch lines appear as singularities of $h_\alpha$ or $h_\beta$, of which both position and strength are directly indicated by the boundary conditions, and a method of dealing with them is described in § 2·3 below. Lines of discontinuity of the velocity gradient, or of higher derivatives of the velocity components, are similarly indicated by the boundary conditions and located on fixed lines $\alpha =$ const. or $\beta =$ const. in the characteristic plane. They need attention only if the boundary conditions are expanded in power series, in which case the solution must be pieced together from the solutions in subregions in which the boundary conditions are analytic (see, for example, § 3·1 below). It may therefore be assumed in the following (except in § 2·3) that $h_\alpha$ and $h_\beta$ are everywhere bounded and are piecewise analytic† functions of $\alpha$ and $\beta$.

Differentiation of (5) and (6), with use of (1) to (4), yields the 'focusing equations',

$$\partial h_\beta/\partial\alpha = m(h_\beta\cos 2\mu - h_\alpha), \tag{9}$$

$$\partial h_\alpha/\partial\beta = m(h_\beta - h_\alpha\cos 2\mu), \tag{10}$$

where
$$m(t) = \tfrac{1}{2}(1-\mathrm{d}\mu/\mathrm{d}t)\operatorname{cosec} 2\mu = (\gamma+1)/(8\sin\mu\cos^3\mu). \tag{11}$$

* By inference from the components of acceleration in the analogous problem treated by Meyer (1952).

† This term will be used here to indicate the existence of continuous partial derivatives of all orders.



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In contrast to (5) and (6), the coefficients in (9) and (10) are known functions of $\alpha$ and $\beta$, in fact, of $(\alpha-\beta)$. Moreover, it is shown in Meyer (1949) that the function

$$f(\mu) = [\{(\gamma-\cos 2\mu)^{\gamma}(\sin\mu)^{-\gamma-1}\}^{1/(\gamma-1)}\sec\mu]^{\frac{1}{2}} \tag{12}$$

(figure 1) plays an important role in the solutions of our equations, and use can be made of this by introducing yet another set of dependent variables,

$$U = h_{\alpha}/f(\mu), \quad V = h_{\beta}/f(\mu), \tag{13}$$

for which (9) and (10) reduce to

$$\partial U/\partial\beta = mV \tag{14}$$

$$\partial V/\partial\alpha = -mU. \tag{15}$$

Alternatively, the variables

$$Z = U+V, \quad Y = U-V \tag{16}$$

may be employed, for which (14) and (15) become

$$\partial^2 Z/\partial\alpha\,\partial\beta + \phi(\alpha-\beta)\,Z = 0, \tag{17}$$

$$\partial^2 Y/\partial\alpha\,\partial\beta + \psi(\alpha-\beta)\,Y = 0, \tag{18}$$

with

$$\phi(\alpha-\beta) = m^2 - \tfrac{1}{2}\mathrm{d}m/\mathrm{d}t = m^2[1+\{2-1/(m\sin 2\mu)\}(3-4\cos^2\mu)], \tag{19}$$

$$\psi(\alpha-\beta) = m^2 + \tfrac{1}{2}\mathrm{d}m/\mathrm{d}t = m^2[1-\{2-1/(m\sin 2\mu)\}(3-4\cos^2\mu)]. \tag{20}$$

It is often convenient to solve both (17) and (18) in order to find $U$ and $V$ from (16), but it is not necessary. For instance, if (17) only is solved, $U$ can be found by quadrature from

$$\partial(bU)/\partial\beta = Z\partial b/\partial\beta,$$

or $V$ from

$$\partial(bV)/\partial\beta = b\,\partial Z/\partial\beta,$$

by (14) and (16), where

$$b = \exp\left\{-2\int^{t} m(t')\,\mathrm{d}t'\right\} = \left[\left\{\frac{\nu+\cos\mu}{\nu-\cos\mu}(\sin\mu)^{\gamma+1}\right\}^{1/(\gamma-1)}\sec\mu\right]^{\frac{1}{2}},$$

with $\nu^2 = \frac{1}{2}(\gamma+1)$.

In any case, $h_{\alpha}$ and $h_{\beta}$ are found from (13), and from them the physical and mathematical structure of the solution can be deduced. The velocity gradient, for instance, is given by (7) and (8). For points of particular interest in the flow plane, the co-ordinates can finally be obtained by quadrature from (5) or (6).

### 2·1. The Riemann functions

For equations (17) and (18), respectively, the Riemann functions $R(\xi,\eta;\alpha,\beta)$ and $S(\xi,\eta;\alpha,\beta)$ satisfy the equations (Courant & Hilbert 1937)

$$R_{\xi\eta} + \phi(\xi-\eta)\,R = 0, \tag{21}$$

$$S_{\xi\eta} + \psi(\xi-\eta)\,S = 0 \tag{22}$$

(suffixes are now used to denote partial differentiation with respect to characteristic variables), with boundary conditions

$$R(\alpha,\eta;\alpha,\beta) = R(\xi,\beta;\alpha,\beta) = 1, \tag{23}$$

$$S(\alpha,\eta;\alpha,\beta) = S(\xi,\beta;\alpha,\beta) = 1. \tag{24}$$

The difficulties in solving these equations arise entirely from the awkward nature of the dependence of $\phi$ and $\psi$ on $(\xi-\eta)$ (see equations (19), (20), (11) and (1) to (4)). Figure 1 shows a graph of $\phi$, $\psi$, $f(\mu)$ and $t$ against the Mach number $M$ for $\gamma = 1{\cdot}4$. Both $\phi$ and $\psi$ are singular at sonic speed and at vacuum speed, but show a relatively slow variation with Mach number at medium supersonic speeds, and $\phi$ has a minimum at about $M = 2{\cdot}5$. Accordingly, attention will first be paid to the medium Mach number range.

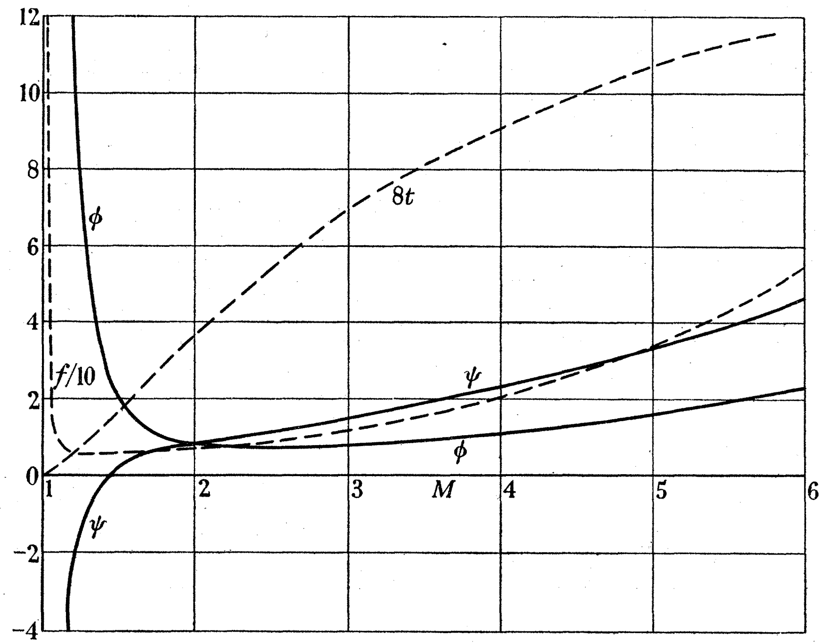


FIGURE 1

In this range we choose a base point $(\alpha_0, \beta_0)$, put

$$\alpha' = \alpha - \alpha_0, \quad \beta' = \beta - \beta_0, \quad \xi' = \xi - \alpha_0, \quad \eta' = \eta - \beta_0,$$

consider $R$ as a function of the dashed variables, and rewrite (21) in the form

$$R_{\xi'\eta'} + \phi_0 R = [\phi_0 - \phi(\xi' - \eta' + \alpha_0 - \beta_0)]\, R, \tag{25}$$

where $\phi_0 = \phi(\alpha_0 - \beta_0)$. If the right-hand side of the equation (25) is now treated as a known function and the equation is solved by Riemann's method in terms of the appropriate Riemann function, $r(x, y; \xi', \eta')$, an integral equation,

$$R(\xi', \eta'; \alpha', \beta') = r(\xi', \eta'; \alpha', \beta') - \int_{\alpha'}^{\xi'}\int_{\beta'}^{\eta'} [\phi(x - y + \alpha_0 - \beta_0) - \phi_0]\, R(x, y; \alpha', \beta')\, r(x, y; \xi', \eta')\, dx\, dy, \tag{26}$$

is obtained for $R$. The Riemann function $r$ is

$$r(x, y; \xi', \eta') = J_0(2[\phi_0(x - \xi')(y - \eta')]^{\frac{1}{2}}) \tag{27}$$

(Courant & Hilbert 1937; $J_0$ denotes the Bessel function of zero order), and constitutes a convenient starting point for an iteration procedure to determine $R$. Indeed, if

$$R_n(\xi', \eta'; \alpha', \beta') = -\int_{\alpha'}^{\xi'}\int_{\beta'}^{\eta'} [\phi - \phi_0]\, r(x, y; \xi', \eta')\, R_{n-1}(x, y; \alpha', \beta')\, dx\, dy, \tag{28}$$

with $R_0 = r$, and if $\sum_{n=0}^{\infty} R_n$ exists, then this series satisfies (21) and (23). A sufficient condition for the convergence of the series can be found by Picard's method (Courant & Hilbert 1937).

A corresponding solution for $S$ is obtained by replacing $\phi$ by $\psi$.

2·1·1. In order to obtain a solution that lends itself more conveniently to numerical handling, in many cases, replace $\phi$ in (28) by its power-series expansion about the base point,

$$\phi(\xi-\eta) = \sum_{k=0}^{\infty} \phi_k(\alpha_0-\beta_0)(\xi'-\eta')^k, \tag{29}$$

and $r$ by the power-series expansion of the Bessel function (27) in terms of its argument. This leads to the approximation

$$R'_N = \sum_{n=0}^{N} R_n$$

being obtained also in form of a power series, of which only those terms need be retained that recur unchanged in $R'_{N+1}$. Thus if $[R_n]^{(k)}$ denotes all those terms in the power series for $R_n$ which are at most of degree $k$ in $(\xi'-\eta')$, $(\alpha'-\beta')$, $(\xi'-\alpha')$ and $(\eta'-\beta')$, then we take as the $N$th approximation for the Riemann function $R$ the polynomial

$$R'_N = \sum_{n=0}^{N} [R_n]^{(3N+2)}, \tag{30}$$

where $R_n$ is given by (28).

The convergence of this scheme of successive approximations can be established by a method similar to Picard's. The series (29) will converge for $|\xi'-\eta'|<\delta$, where $\delta$ is a positive constant depending on the choice of the base point, and hence the coefficients $\lambda_k$ of the series for $r\phi$,

$$\phi(\xi'-\eta') \sum_{n=0}^{\infty} [-\phi_0(\xi'-x)(\eta'-y)]^n (n!)^{-2} = \sum_{k=0}^{\infty} \lambda_k(x, y; \xi'+\eta'; \alpha_0-\beta_0)(\xi'-\eta')^k,$$

will satisfy $|\lambda_k|\delta^k < K(\alpha_0-\beta_0)$, with $K$ independent of $k$. A sufficient condition for the convergence of $R'_N$ to $R$ can then be shown to be

$$(\epsilon/\delta)^2 + |\xi'-\alpha'|\,|\eta'-\beta'|\,(1+\epsilon/\delta)\,K < 1,$$

where $\epsilon$ denotes the largest value of $|x-y|$ occurring in the characteristic rectangle over which the integral is taken in (28). A necessary condition is not established here, but it may be noted that $R$ is analytic in all its arguments, provided only $\xi-\eta$ and $\alpha-\beta$ do not attain values corresponding to sonic or vacuum conditions. For the equations (17) and (18) are self-adjoint, and hence $R$ and $S$ (i) depend symmetrically on $\xi$, $\eta$ and $\alpha$, $\beta$ (Courant & Hilbert 1937) and (ii) represent the solution of a particular flow problem with analytic boundary conditions and, like the corresponding length parameters, they are therefore analytic in $\xi$ and $\eta$.

To obtain explicit expressions for the polynomials $R'_N$, note that they depend only on the variables

$$\left.\begin{aligned} \nu &= \xi'-\alpha' = \xi-\alpha, \\ \rho &= \eta'-\beta' = \eta-\beta, \\ \sigma &= \alpha'-\beta' = \alpha-\beta-\alpha_0+\beta_0, \end{aligned}\right\} \tag{31}$$

and (the last of these)

since

$$R_0(\xi', \eta'; \alpha', \beta') = J_0(2[\phi_0 \nu\rho]^{\frac{1}{2}}),$$

by (27) and (29), and since, if $R_{n-1}(\xi', \eta'; \alpha', \beta') = R_{n-1}(\nu, \rho, \sigma)$, (28) can be written

$$R_n = -\nu\rho \int_0^1\int_0^1 J_0(2[\phi_0 \nu\rho(1-u)(1-w)]^{\frac{1}{2}})\, R_{n-1}(\nu u, \rho w, \sigma) \sum_{k=1}^{\infty} \phi_k(\sigma+\nu u-\rho w)^k \,du\,dw$$
$$= R_n(\nu, \rho, \sigma).$$

By expanding the Bessel function in power series and performing the integrations, we obtain for the polynomial $[R]^n$ that differs from the Riemann function $R$ only by terms of higher order than $n$ in $\nu$, $\rho$ and $\sigma$, the expression

$$[R]^n = \sum_{p=0}^{n} \sum_{q=0}^{s} R_q^{(p)}, \tag{32}$$

where $s$ is the greatest integer such that $3s \leqslant p$:

$$R_0^{(0)} = 1,$$

$$\left.\begin{aligned} R_0^{(2m+1)} &= 0, \\ R_0^{(2m)} &= (-1)^m (\phi_0 \nu\rho)^m (m!)^{-2}, \end{aligned}\right\} \text{ for integral } m,$$

$$R_1^{(p)} = 0 \quad \text{for } p \leqslant 2,$$

$$R_1^{(3)} = -\phi_1 \nu\rho(\sigma + \tfrac{1}{2}(\nu-\rho)),$$

$$R_1^{(4)} = -\phi_2 \nu\rho[\tfrac{1}{3}(\nu^2+\rho^2) - \tfrac{1}{2}\nu\rho + \sigma(\nu-\rho) + \sigma^2],$$

$$R_1^{(5)} = \tfrac{1}{4}\phi_0\phi_1 \nu^2\rho^2(\nu-\rho+2\sigma) + (\tfrac{1}{20}\phi_3)\,[(\nu-\rho+\sigma)^5 - (\sigma+\nu)^5 - (\sigma-\rho)^5 + \sigma^5],$$

$$R_1^{(6)} = \tfrac{1}{2}\phi_0 \nu^2\rho^2[\tfrac{1}{3}(\nu^2+\rho^2) - \tfrac{5}{9}\nu\rho + \sigma(\nu-\rho) + \sigma^2] + (\tfrac{1}{30}\phi_4)\,[(\nu-\rho+\sigma)^6 - (\sigma+\nu)^6 - (\sigma-\rho)^6 + \sigma^6],$$

etc., and

$$R_2^{(p)} = 0 \quad \text{for} \quad p \leqslant 5,$$

$$R_2^{(6)} = (\tfrac{1}{4}\phi_1 \nu\rho)^2\,[\tfrac{1}{4}(\nu^2+\rho^2) - \tfrac{4}{9}\nu\rho + \sigma(\nu-\rho) + \sigma^2],$$

etc.

The corresponding approximations for the Riemann function $S$ are obtained by replacing $\phi_k$ by the coefficient $\psi_k$ of the power series of $\psi$ about the base point,

$$\psi(\xi-\eta) = \sum_{k=0}^{\infty} \psi_k(\alpha_0-\beta_0)\,(\xi'-\eta')^k. \tag{33}$$

Tables of $t$, $\mu$, $f(\mu)$, $\phi$ and $\psi$, for $\gamma = 1{\cdot}4$ and $1{\cdot}3$, are given in the Appendix.

Note that the coefficients in the approximations $[R]^{(n)}$ and $[S]^{(n)}$ for the Riemann functions depend only on the coefficients of the series (29) and (33), respectively. It follows that they depend only on $(\alpha_0-\beta_0)$ and hence only on the choice of the base Mach number of the approximation.

It will be seen below that even the trivial approximation, $[R]^{(0)} \equiv [S]^{(0)} \equiv 1$, yields a solution that is at least equivalent to a uniform first-order theory of the conventional equations of the flow (§ 3·2·3). For some problems it yields a solution of accuracy greatly superior to that given by linearized theory (§ 3·0·1). The expressions here given explicitly therefore suffice for the treatment of problems involving relatively large variations of pressure and Mach number. The rapid convergence of the partial sums (32) to $R$ depends, nevertheless, to some degree on the rapid convergence of the series (29) for $\phi$, and when that convergence is slow, then the iteration (28) may be more convenient.

The results obtained indicate more clearly the extent of the 'medium Mach number range', to which they apply. Two points of view need be distinguished. In principle, the restrictions are only $M > 1$ and $1/M > 0$, for these conditions ensure convergence, provided the interval of Mach numbers to be covered in a problem is sufficiently small. For numerical purposes, on the other hand, the effort required to compute the solution to a given accuracy is more relevant. If the centre of the interval over which the Mach number varies in any given problem be supposed fixed, then the effort increases with the extent of the interval; if the extent be supposed fixed, then the effort is the greater the nearer one of the extreme speeds occurring comes to either sonic or vacuum speed.

### 2·2. Double power series

From (30), the Riemann function is obtained in the form of a power series. This suggests that it may sometimes be profitable to obtain the solution of (14) and (15) directly in form of such a series. Like $\phi$ and $\psi$, $m$ may be expanded in a series

$$m(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j (i!j!)^{-1} m_{i+j} (\alpha-\alpha_0)^i (\beta-\beta_0)^j, \tag{34}$$

and if we write

$$U = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} u_{r,s} (\alpha-\alpha_0)^r (\beta-\beta_0)^s, \tag{35}$$

$$V = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} v_{r,s} (\alpha-\alpha_0)^r (\beta-\beta_0)^s, \tag{36}$$

and substitute these series in (14) and (15), the recurrence relations

$$(s+1)\, u_{r,s+1} = \sum_{i=0}^{r} \sum_{j=0}^{s} (-1)^j (i!j!)^{-1} m_{i+j} v_{r-i,s-j} \tag{37}$$

and

$$(r+1)\, v_{r+1,s} = -\sum_{i=0}^{r} \sum_{j=0}^{s} (-1)^j (i!j!)^{-1} m_{i+j} u_{r-i,s-j} \tag{38}$$

are obtained.

Next, the boundary conditions have to be expanded in power series to supplement (37) and (38) in order to form a system of equations from which the coefficients $u_{r,s}$ and $v_{r,s}$ can be determined. This step is best discussed at the instance of examples, and the discussion is deferred to §§ 3·1·1 and 3·3 below.

The convergence of the series (35) and (36) is assured, under suitable assumptions on the boundary conditions, since the same series will be obtained by Riemann's method when both the Riemann functions and the boundary conditions are employed in their power-series forms.

### 2·3. Branch lines

The singularites of $U$ and $V$ are called branch lines. For instance, suppose that $U$ and $V$ are prescribed respectively on $AE$, $EB$ and $AC$ (figure 2) as continuous functions, except that $U(\alpha,\beta_1) \sim (\alpha-\alpha_b)^{-\frac{1}{2}}$ near $E$. Then $EF$ is a ('single') branch line (Meyer 1949).

To reduce the problem to one that can be treated by the methods so far described, put

$$U_1(\alpha,\beta) = \sum_{n=0}^{\infty} c_n(\beta)\, (\alpha-\alpha_b)^{n-\frac{1}{2}}, \tag{39}$$

expand $m$ in a Taylor series

$$m(t) = \sum_{s=0}^{\infty} m_s(\beta)\, (\alpha-\alpha_b)^s, \tag{40}$$

and substitute into
$$V_1 = -\int_{\alpha_b}^{\alpha} mU_1\,\mathrm{d}\alpha, \tag{41}$$
$$U_1 = \int_{\beta_1}^{\beta} mV_1\,\mathrm{d}\beta, \tag{42}$$
in order to satisfy (15) and (14). There results
$$V_1(\alpha,\beta) = \sum_{n=1}^{\infty} d_n(\beta)\,(\alpha-\alpha_b)^{n-\frac{1}{2}}, \tag{43}$$
and
$$\left.\begin{aligned} d_i(\beta) &= \frac{-2}{2i-1}\sum_{k=0}^{i-1} c_k m_{i-k-1} \\ c_i(\beta) &= \int_{\beta_1}^{\beta}\sum_{p=1}^{i} d_p(\beta')\,m_{i-p}(\beta')\,\mathrm{d}\beta' \end{aligned}\right\}\quad (i\geqslant 1),^*$$
$$c_0 = \text{const.}$$

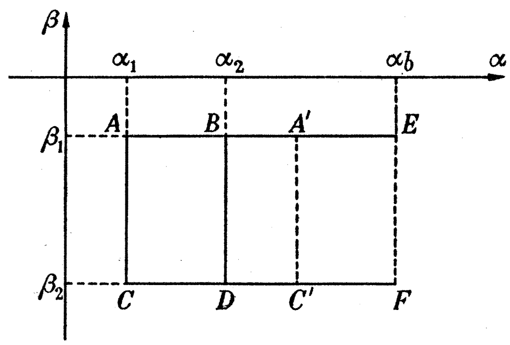


FIGURE 2

If, in addition to our choice of $\beta_1$ as limit of integration, we choose
$$c_0 = \lim_{\alpha=\alpha_b}\,[(\alpha-\alpha_b)^{\frac{1}{2}}\,U(\alpha,\beta_1)],$$
then $U_1(\alpha,\beta_1)$ is just the singular part of $U(\alpha,\beta_1)$. The choice of $\alpha_b$ as limit of integration is imposed by our assumption that the series (39) for $U_1$ contains no integral powers of $(\alpha-\alpha_b)$, and it implies
$$V_1(\alpha_b,\beta) = 0. \tag{44}$$

The series (39) and (43) can be derived term by term from (41) and (42) by an iteration process, the convergence of which can be shown by a modification of Picard's method. If the series (40) converges for $|\alpha-\alpha_b|<\delta_1$, then $|m_s(\beta')|\,\delta_1^s<K_1(\beta')$ for $\beta\leqslant\beta'\leqslant\beta_1$, and a sufficient condition for convergence of (39) and (43) is $|\alpha-\alpha_b|<(K_1^2\,|\beta-\beta_1|+2/\delta_1)^{-1}$. Hence $(U-U_1)$ and $(V-V_1)$ satisfy (14) and (15) with regular boundary conditions on $AE$, $BE$ and (at least) some line $A'C'$ between $AC$ and $EF$, and so these functions can be found in $AEFC$ and $BEFD$ by the methods described in the preceding sections.

It may be worth noting that, in our particular solution $(U_1, V_1)$, $EF$ is not only a branch line, but also a limit line (Meyer 1949), by (44); indeed, $EF$ is mapped into a point in the physical plane and our particular solution is a 'centred wave', even if not a simple centred

* The calculation of the coefficients $c_i$, $d_i$ is facilitated by noting that $m_s(\beta) = (-1)^s\,(s!)^{-1}\,\mathrm{d}^s m_0/\mathrm{d}\beta^s$, since $t = \frac{1}{2}(\alpha-\beta)$.

wave (Prandtl–Meyer expansion), for the Mach lines, $\beta = \text{const.}$, are straight only locally at the centre, and curved elsewhere. With the exception of the line $EF$, however, $U_1$ and $V_1$ are analytic in $\alpha$ and $\beta$, since they represent the solution of a particular flow problem with boundary conditions that are analytic, except for $U_1$ at $E$.

Other cases of branch lines can be dealt with similarly. For instance, for a 'double' branch line (Meyer 1949) the corresponding series to start from would be

$$U_2 = \sum_{n=0}^{\infty} c_n(\beta)\,(\alpha-\alpha_b)^{n-\frac{2}{3}},$$

and the same procedure would lead to the construction of the solution for the boundary conditions

$$U_2(\alpha,\beta_1) = c_0(\alpha-\alpha_b)^{-\frac{2}{3}},$$
$$V_2(\alpha_b,\beta) = 0.$$

The method just described can serve as an alternative to the methods based on the Riemann function and on the double power series, both for the construction of particular solutions and of solutions for definite boundary conditions, especially if a more general choice of the lower limits of integration in (41) and (42) is admitted.

### 2·4. Hypersonic flow

Since both $\phi$ and $\psi$ are singular for $M = \infty$, the convergence of the series (29) and (33) is too slow if the base Mach number is too large, and it is more convenient to expand about a base point for which $1/M = 0$. If $\alpha'$, $\beta'$ are the co-ordinates in the characteristic plane referred to such a base point as origin, then

$$\phi(\alpha'-\beta') = \sum_{k=-1}^{\infty} \phi'_k(\alpha'-\beta')^{2k}$$

and

$$\psi(\alpha'-\beta') = \sum_{k=-1}^{\infty} \psi'_k(\alpha'-\beta')^{2k},$$

and this suggests that equation (21) be rewritten in the form

$$R_{\xi'\eta'} + \phi'_{-1}(\xi'-\eta')^{-2}R = -R\sum_{k=0}^{\infty}\phi'_k(\xi'-\eta')^{2k}.$$

If the right-hand side of this equation is treated as if it were known, then it is again an equation for which the Riemann function, $r'$, is known, and by its help the equation can be transformed into an integral equation similar to (26).

The Riemann function $r'$ is (Riemann 1860)

$$r'(x,y;\xi',\eta') = \left[\frac{(x-y)\,(\xi'-\eta')}{(x-\eta')\,(\xi'-y)}\right]^{\lambda} F\left\{\lambda,\lambda,1,\frac{(x-\xi')\,(y-\eta')}{(\eta'-x)\,(\xi'-y)}\right\},$$

where $F$ denotes the hypergeometric function and $\lambda$ is either root of

$$\lambda(\lambda-1) = \phi'_{-1}.$$

If $\phi$ and $\psi$ are expanded, by the help of (19), (20) and (11), it is found that

$$\phi'_{-1} = (\gamma+1)\,(3-\gamma)/[4(\gamma-1)^2] \quad\text{and}\quad \psi'_{-1} = (\gamma+1)\,(3\gamma-1)/[4(\gamma-1)^2],$$

and it follows that

$$\lambda(\phi'_{-1}) = \tfrac{1}{2}(\gamma+1)/(\gamma-1) \quad \text{or} \quad -\tfrac{1}{2}(3-\gamma)/(\gamma-1)$$

and

$$\lambda(\psi'_{-1}) = \tfrac{1}{2}(3\gamma-1)/(\gamma-1) \quad \text{or} \quad -\tfrac{1}{2}(\gamma+1)/(\gamma-1).$$

Now if $\lambda$ is a negative integer, the power-series expansion of the hypergeometric function terminates after a finite number of terms and hence in that case the analysis is greatly simplified. The respective values of $\gamma$ for which $\lambda$ is a negative integer, $-n$, are

$$\gamma(\phi'_{-1}) = (2n+3)/(2n+1) \quad (n = 0, 1, 2, \ldots),$$

and

$$\gamma(\psi'_{-1}) = (2n+1)/(2n-1) \quad (n = 1, 2, \ldots),$$

in fact, just those values of $\gamma$ which play a similar role in the theory of one-dimensional, unsteady motion of a perfect gas (Love & Pidduck 1922). This is not unexpected, in view of the work of Hayes (1947) and Goldsworthy (1952) on the analogy between one-dimensional unsteady motion and hypersonic flow.

The interest of the present results lies in that they suggest a possibility of obtaining better approximations for hypersonic flows than those known to date. If negative values of $\lambda$ are taken, the hypergeometric functions remain bounded as the vacuum point is approached, for then their argument approaches unity from below along the real axis. Moreover, the existence of Hayes's analogy suggests that $r'$ is indeed a suitable starting point for a convergent iteration scheme yielding the Riemann function, $R$. A detailed examination will, however, be deferred to a later date.

### 2·5. *Near-sonic flow*

The functions $\phi$ and $\psi$ are also singular for $M = 1$ and their expansions about a base point with this Mach number are of the form

$$\phi(\alpha-\beta) = (\alpha-\beta)^{-2} \sum_{k=0}^{\infty} \phi''_k(\alpha-\beta)^{\frac{2}{3}k} \tag{45}$$

and

$$\psi(\alpha-\beta) = (\alpha-\beta)^{-2} \sum_{k=0}^{\infty} \psi''_k(\alpha-\beta)^{\frac{2}{3}k}, \tag{46}$$

with $\phi''_0 = \frac{7}{36}$ and $\psi''_0 = -\frac{5}{36}$, by (19), (20), (11) and (1) to (4), independently of the value of $\gamma$. Integral equations for $R$ and $S$ can therefore be obtained just as in §§ 2·1 and 2·4, the auxiliary Riemann functions being now

$$r'' = \left[\frac{(x-y)\,(\xi-\eta)}{(\xi-y)\,(x-\eta)}\right]^{-\frac{1}{6}} F\left\{-\frac{1}{6}, -\frac{1}{6}, 1, \frac{(\xi-x)\,(y-\eta)}{(\xi-y)\,(x-\eta)}\right\}$$

and

$$s'' = \left[\frac{(x-y)\,(\xi-\eta)}{(\xi-y)\,(x-\eta)}\right]^{\frac{1}{6}} F\left\{\frac{1}{6}, \frac{1}{6}, 1, \frac{(\xi-x)\,(y-\eta)}{(\xi-y)\,(x-\eta)}\right\}.$$

These hypergeometric functions remain bounded as $M \to 1$, but the hypergeometric series do not degenerate. Moreover, the convergence of an iteration process like the one employed in § 2·1·1 must now be much slower, on account of the form of the series (45) and (46).

Despite these difficulties, the present results suggest a possibility of obtaining higher approximations to the first-order, transonic theory based on Tricomi's equation. The usefulness of the present approach is, however, seriously impaired by the restriction to supersonic flow implicit in the adoption of characteristic variables.

## 3. The first wave interaction in a supersonic jet

When a two-dimensional jet issues from a perfect, supersonic nozzle into a container at a pressure lower than the ultimate pressure in the nozzle, the expansion at the lips of the nozzle ($A$ and $A'$ in figure 3) produces two centred simple waves ($ABC$ and $A'BC'$). The first wave-interaction problem in the jet is therefore the interaction of two symmetrical, centred simple waves; it can also be interpreted as the reflexion of a centred simple wave from a straight streamline.

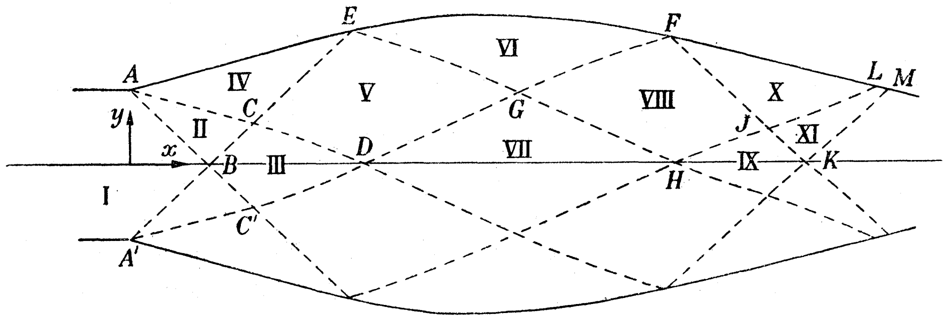


Figure 3. Flow plane.

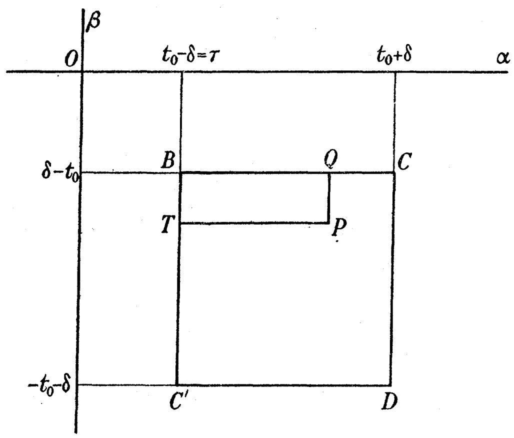


Figure 4. Characteristic plane.

Let $\delta$ denote the angle through which the stream is deflected at the centre $A$. Then $\theta(B) = 0$, $\theta(C) = \delta$, and since $\beta(C) = \beta(B)$ and $\beta(B) = -\alpha(B) = -t(B)$, $t(C) = \delta + t(B)$. Again, $\theta(D) = 0$ and $\alpha(D) = \alpha(C)$, so $t(D) = t(B) + 2\delta$. The mapping of the interaction region $BCDC'$ into the characteristic plane is therefore the square shown in figure 4. Corresponding points in figures 3 and 4 are denoted by the same letter.

Let the centre of the square be chosen as the base point of the approximation. The value of $t$ and $\alpha$ at $B$ is then

$$\tau = t_0 - \delta,$$

and the ultimate Mach angle in the nozzle (half the angle $ABA'$) is $\mu_N = \mu(\tau)$. Moreover, choose half the width $AA'$ of the nozzle mouth as unit of length.

To derive the boundary condition on $BC$ ($\beta = -\tau$), the simple wave $ABC$ (figure 3) needs to be considered briefly. Here $\beta \equiv$ const., so $1/h_\beta = 0$ and from (10), $\partial h_\alpha / h_\beta \partial\beta = m$, and since $h_\alpha = 0$ at $A$,

$$h_\alpha(R) = m\int_A^R h_\beta \, d\beta = ml(\alpha),$$


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where $R$ is any point on $BC$ and the integral is taken along the straight Mach line $AR$. It is shown by Meyer & Holt (1950) that the length, $l(\alpha)$, of the straight Mach line segment between two fixed, curved Mach lines in a simple wave varies proportionally to $f(\mu)$, and since the length of the segment $AB$ is $\operatorname{cosec}\mu_N = M_N$, the ultimate Mach number in the nozzle,

$$h_\alpha(R) = cmf(\mu), \quad \text{where} \quad c = M_N/f(\mu_N) \tag{47}$$

and

$$U(\alpha, -\tau) = cm. \tag{48}$$

It follows by (1), (2) and (15) that $U_\alpha = \frac{1}{2}c\,\mathrm{d}m/\mathrm{d}t$ and $V_\alpha = -cm^2$ on $BC$, and by (16), (19) and (20),

$$\left.\begin{aligned} Z_\alpha(\alpha, -\tau) &= -c\phi(\alpha+\tau), \\ Y_\alpha(\alpha, -\tau) &= \phantom{-}c\psi(\alpha+\tau). \end{aligned}\right\} \tag{49}$$

The boundary conditions on $BC'$ can be deduced similarly. They are

$$h_\beta = -cmf(\mu), \tag{50}$$

$$\left.\begin{aligned} Z_\beta(\tau, \beta) &= -c\phi(\tau-\beta), \\ Y_\beta(\tau, \beta) &= -c\psi(\tau-\beta). \end{aligned}\right\} \tag{51}$$

By (47), (50), (13) and (16), the value of $Y$ at $B$ is

$$Y(\tau, -\tau) = 2cm(\tau). \tag{52}$$

The solution for $Y$ at an arbitrary point $P$ in the square $BCDC'$ (figure 4) can now be obtained by the help of Riemann's method. That is, multiply (18) by $S$ and (22) by $Y$, subtract, integrate the resulting identity with respect to $\xi$ and $\eta$ over the rectangle $BTPQ$ (figure 4), and integrate by parts to obtain

$$\int (SY_\xi l_\eta - YS_\eta l_\xi)\,\mathrm{d}s = 0,$$

where the integral is taken anti-clockwise round the contour $BTPQ$ and $l_\xi$, $l_\eta$ denote respectively the components of the outward unit normal in the directions of $\xi$ and $\eta$ increasing. On account of the boundary conditions (24) for $S$, this reduces to

$$-\int_Q^B SY_\xi\,\mathrm{d}\xi - \int_B^T YS_\eta\,\mathrm{d}\eta - \int_T^P Y_\xi\,\mathrm{d}\xi = 0.$$

Upon integrating by parts in the second integral and then substituting the boundary conditions (49), (51) and (52) for $Y$,

$$Y(P) = 2cm(B)\,S(B) - c\int_B^T \psi S\,\mathrm{d}\eta - c\int_Q^B \psi S\,\mathrm{d}\xi$$

is obtained, and by (22) and (24), this reduces to

$$Y(P) = 2cm(B)\,S(B) - cS_\xi(B) + cS_\eta(B),$$

i.e.

$$\begin{aligned} Y(\alpha,\beta) &= 2cm(\tau)\,S(\tau, -\tau;\, \alpha,\beta) + c[S_\eta(\tau, -\tau;\, \alpha,\beta) - S_\xi(\tau, -\tau;\, \alpha,\beta)] \\ &= 2cm(\tau)\,S(\tau, -\tau;\, \alpha,\beta) - c\frac{\mathrm{d}}{\mathrm{d}\tau}S(\tau, -\tau;\, \alpha,\beta). \end{aligned} \tag{53}$$

Similarly,

$$Z(\alpha,\beta) = c[R_\alpha(\tau, -\tau;\, \alpha,\beta) + R_\beta(\tau, -\tau;\, \alpha,\beta)].$$

The solutions of the (exact) equations (17) and (18) for this problem are thus expressed directly in terms of the Riemann functions, without even a quadrature.

3·0·1. To find the pressure distribution on the axis of symmetry, note that

$$\frac{p}{p_N} = \left[\frac{\sin^2\mu}{\sin^2\mu_N}\,\frac{\gamma-1+2\sin^2\mu_N}{\gamma-1+2\sin^2\mu}\right]^{\gamma/(\gamma-1)}, \tag{54}$$

by Bernoulli's equation, where $p$ is the pressure and $\mu$ the Mach angle. On the axis of symmetry, $\theta = \frac{1}{2}(\alpha+\beta) = 0$ and hence $t = \alpha$ and

$$\mathrm{d}x/\mathrm{d}t = (h_\alpha - h_\beta)\cos\mu, \tag{55}$$

by (1), (2), (5) and (6). It follows from (13) and (16) that

$$x(t) = x(\tau) + \int_\tau^t Y(t, -t)\, f(\mu) \cos\mu \,\mathrm{d}t, \tag{56}$$

$x(\tau)$ being the abscissa of $B$ (figure 4).

These results have been applied to the problem of a jet of gas (with $\gamma = 1\cdot3$) issuing from a perfect nozzle at Mach number 1·5 into an atmosphere at a pressure equal to half the initial pressure of the jet. This example was chosen to make possible a comparison with the results obtained by Pack (1948) by Massau's numerical method* of integration of the characteristic equations. It also provided a practical test of our method in a case where the iteration procedure devised for the medium Mach number range was not *a priori* certain to converge rapidly.

For $\gamma = 1\cdot3$, the sonic singularity of $\psi$ has a marked influence on the behaviour of this function even at Mach numbers near 1·5, and the Taylor series of $\psi$ near the base point was found not to converge rapidly to $\psi$ over the whole range of the problem. An approximating polynomial, $\Psi$, was therefore fitted to $\psi$ by the method of least squares and, with $\Psi$ employed in the place of $\psi$, the iteration (28) was carried to an accuracy slightly exceeding that obtainable from the approximation given explicitly in § 2·1·1.

The results of the computation† are shown in table 1 and figure 5. An upper bound for the error, obtained directly from the integral equation for the Riemann function, guarantees that the error is less than 0·002 in the pressure ratio. Moreover, with the approximate solution thus far known, a close estimate of the error‡ can be obtained by the help of the same integral equation and of the known function, $\psi - \Psi$. This indicates an error of less than 0·0002 in the pressure ratio. Pack's (1948) results are also shown in figure 5, and it appears§ that his own estimate of the error in his results is definitely conservative.

The accurate results for the pressure distribution provide an opportunity for a numerical assessment of rougher approximations. The prediction of linearized theory is only

$$x(t) = \text{const.} = x(\tau) \tag{57}$$

(figure 5). A better approximation is obtainable, without any consideration of the wave interaction, from the known, exact solution for the Prandtl–Meyer expansions centred at

* In which it is difficult to estimate the errors.

† Which was carried out by Mr P. Skeat, Aeronautical Research Laboratories, Melbourne.

‡ Though not necessarily an upper bound.

§ The authors are indebted to Professor Pack for the communication of detailed, unpublished numerical results.

the nozzle lips. This yields the pressure distribution on the Mach lines through $B$ (figure 3) and hence the value of $dp/dx$ at $B$. In this way* the tangent at $B$ to the curve of $p(x)$ is found to be

$$x(t) = x(\tau)+\frac{\gamma+1}{2\gamma}\operatorname{cosec} 2\mu_N(1-p/p_N) \tag{58}$$

(figure 5). The same result is, of course, obtained by expanding (54) and (56) in power series with respect to $\mu$ or $t$ and retaining only the linear terms.

TABLE 1

| $x$ | $p/p_N$ | $M$ | $x$ | $p/p_N$ | $M$ |
|---|---|---|---|---|---|
| 1·1180 | 1·0000 | 1·5000 | 2·2 | 0·3773 | 2·1213 |
| 1·2 | 0·9140 | 1·5613 | 2·4 | 0·3282 | 2·2055 |
| 1·4 | 0·7428 | 1·6980 | 2·6 | 0·2883 | 2·2834 |
| 1·6 | 0·6141 | 1·8199 | 2·8 | 0·2555 | 2·3559 |
| 1·8 | 0·5153 | 1·9298 | 3·0 | 0·2282 | 2·4236 |
| 2·0 | 0·4383 | 2·0298 | 3·0338 | 0·2239 | 2·4346 |
| | | | (3·2 | 0·2050 | 2·4871) |

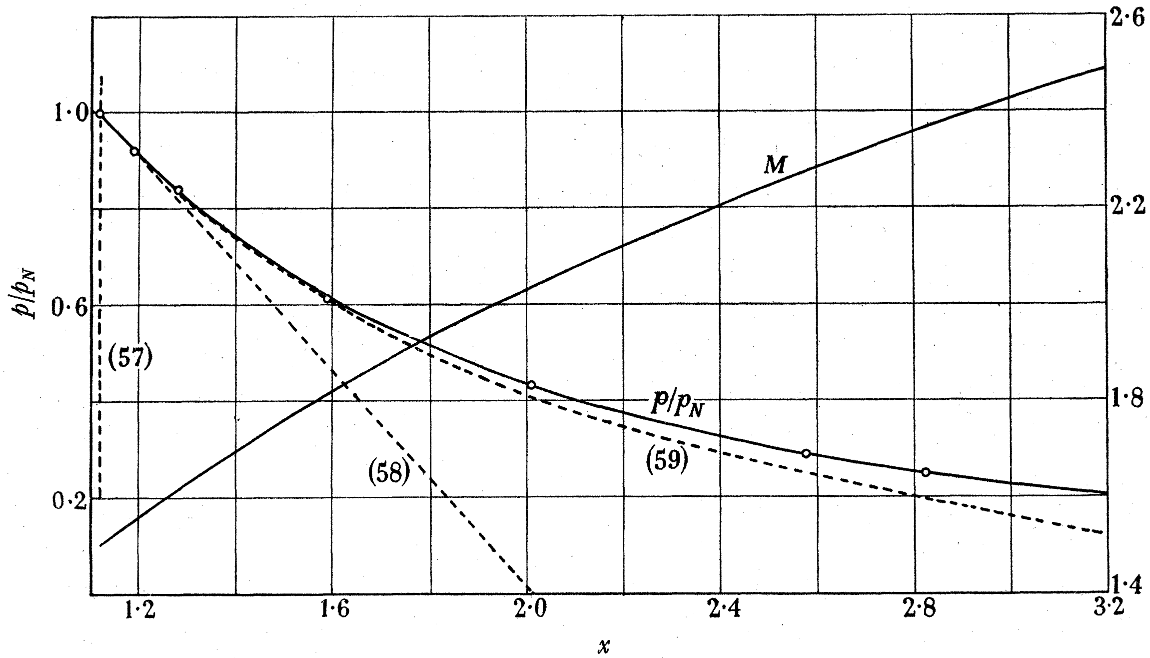


FIGURE 5. The circles represent the points obtained by Pack (1948).

This is not, however, the first approximation suggested by the present theory of the wave interaction. The exact solution is obtained in two stages, first by a transformation of the characteristic equations into the equations (17) and (18) and secondly by the determination of the Riemann functions for these latter equations by successive approximations. A first approximation is thus obtained by using the exact transformations of the first stage in conjunction with the roughest approximation, $S\equiv 1$, for the Riemann function. Then (56) reduces to

$$x(t) = x(\tau)-\frac{\gamma+1}{2\sin^2\mu_N\cos^3\mu_N f(\mu_N)}\int_{\mu_N}^{\mu}\frac{f(\mu')\cos^3\mu'}{\gamma-\cos 2\mu'}\,d\mu', \tag{59}$$

* Or from (55), (13), (16), (56), (53) and from (3) and Bernoulli's equation.

by (53), (47) and (11). The pressure distribution found from (59) and (54) is also shown in figure 5, and it is seen to represent a much better approximation than (58). This supports the view that the exact transformation leading to equations (17) and (18) embodies the main, non-linear elements of the solution so that (17) and (18) represent only a residual problem, and the accurate calculation of the Riemann functions is a matter of relatively less importance. Higher approximations should therefore be sought along the same lines as this first, non-linear approximation. The first approximation to the Riemann function,

$$S_0 = J_0(2[\psi_0(\xi-\alpha)(\eta-\beta)]^{\frac{1}{2}}),$$

was found to give an error less than 0·01 in the pressure ratio.

### 3·1. The first period of a supersonic jet

Linearized theory predicts an exactly periodic structure for a supersonic jet expanding from a perfect nozzle, but experiment and two numerical computations by Pack (1948) show that it is only approximately periodic, that shock waves are present and that these effects result from the interactions of simple waves that occur in the jet. The purpose of this section is to examine analytically the features of the flow for the case when the pressure everywhere in the jet differs but little from the ultimate pressure in the nozzle.

If the technique of § 3 were applied to the second interaction region of the jet, which is adjacent to the jet boundary, an integral equation would be obtained with the Riemann function as kernel. Since a process of iteration or of series expansion would be required for its solution, it appears preferable to resort to the method of double power series from the start. The boundary conditions are firm in the characteristic plane, but they are only piecewise analytic, and hence it is not possible to represent the complete solution by a single pair of power series (35) and (36). It is necessary to subdivide the first period into a number of regions, which are indicated in figure 3 (p. 480), and to deal in turn with the regions numbered I to XI. The solution in the unnumbered regions is obtained from that in the others by symmetry.

In outline, the procedure to be adopted is as follows. In region I the flow is uniform and in region II it is a centred simple wave (Prandtl–Meyer expansion) of which the exact solution is known and has been employed in § 3 to find the distribution, $U_1(\alpha)$, of $U$ on the characteristic $BC$ (figure 3). By the help of this boundary condition, the solution for region III (the first interaction region) is calculated to the desired order of accuracy by the method of double power series,* and so the distribution, $V_1(\beta)$, of $V$ on the characteristic $CD$ is determined. This serves as initial condition for region V, which is a simple wave. The solution for region V can thus be written down and employed to find the distribution, $V_2(\beta)$, of $V$ on $EG$. This, in turn, serves as boundary condition for region VI, the second interaction region, for which the solution is obtained by the method of double power series. In particular, the distribution, $U_2(\alpha)$, of $U$ on $GF$ is calculated, which serves as initial condition for the simple wave of region VIII, and so on until the solution in region XI is calculated. In region IV, VII and X the flow is uniform, and the respective values of $\mu$ and $\theta$ in these regions are obtained by tracing back appropriate characteristics. Region IX is a third interaction region and region XI another simple wave.

* As an alternative to the method of § 3.

3·1·1. For simplicity, let $\alpha_0$ and $\beta_0$ be the values of the characteristic variables in region I,* and let the unit of length in the physical plane be chosen so that the length of $AB$ equals $f(\mu_0)$.* If the stream is deflected at the lip of the nozzle through the angle $\delta$, which is assumed to be small, then $\alpha = \alpha_0 + 2\delta$ in region IV, and on the characteristic $BC$,

$$U = U_1(\alpha) = m(\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta_0)$$
$$= m_0 + m_1(\alpha-\alpha_0) + \tfrac{1}{2}m_2(\alpha-\alpha_0)^2 + O(\delta^3), \quad (60)$$

by (47) and (34). Comparison with (35) shows that $u_{r,0} = m_r/(r!)$ in region III. Moreover, $U+V$ is proportional to the streamline curvature (Meyer 1949), which vanishes on the axis and, if it is noted that $\theta = 0$ in region I and hence $\beta_0 = -\alpha_0$, it follows from (35) and (36) that

$$u_{0,0} = -v_{0,0}, \quad u_{1,0} - u_{0,1} = -(v_{1,0} - v_{0,1}), \quad \text{etc.},$$

and the recurrence relations, (37) and (38), then suffice to calculate the coefficients $u_{r,s}$ and $v_{r,s}$. Finally, the following distribution is obtained for $V$ on $CD$:

$$V_1(\beta) = V(\alpha_0+2\delta, \beta) = -m_0(1+2m_0\delta+4m_1\delta^2) - (\beta-\beta_0)\{m_1+2(m_0^3+m_0m_1)\delta\}$$
$$-\tfrac{1}{2}m_2(\beta-\beta_0)^2 + O(\delta^3). \quad (61)$$

Turning to region V, where $\alpha \equiv \alpha_0 + 2\delta$, the distribution, $V_2(\beta)$, of $V$ on $EG$ is found from

$$V_2(\beta) - V_1(\beta) = -\int mU\,\mathrm{d}\alpha = -\int (m/f(\mu))\,h_\alpha\,\mathrm{d}\alpha = -ml(\beta)/f(\mu),$$

by (15) and (13), since the integration is taken along a straight Mach line $\beta = \text{const.}$, on which $m$ and $\mu$ are constant. The function $l(\beta)$ is the length of this minus Mach line, between the plus Mach lines $CD$ and $EG$, and it varies proportionally to $f(\mu)$ (Meyer & Holt 1950), whence

$$V_2(\beta) = V_1(\beta) - ml(\beta_0)/f(\mu_1), \quad (62)$$

where $l(\beta_0)$ is the length of $CE$ (figure 3) and $\mu_1$ is the Mach angle in region IV. Since the flow is uniform in region IV, the lengths of $CE$ and $CA$ are equal. That of $CA$, moreover, is $f(\mu_1)/f(\mu_0)$ times that of $AB$ (§ 3) and hence (62) reduces to

$$V_2(\beta) = V_1(\beta) - m(\tfrac{1}{2}\alpha_0 + \delta - \tfrac{1}{2}\beta)$$
$$= -2[m_0 + (m_0^2+m_1)\delta + (2m_0m_1+m_2)\delta^2]$$
$$+2(\beta-\beta_0)\{m_1 + (m_2+m_0m_1+m_0^3)\delta\} - m_2(\beta-\beta_0)^2 + O(\delta^3). \quad (63)$$

Thus the boundary conditions for region VI, the second interaction region, become on $EG$

$$V(\alpha_0+2\delta, \beta) = V_2(\beta),$$

and on the jet boundary $EF$, where $\mathrm{d}y/\mathrm{d}x = \tan\theta$ and $p = \text{const.}$ (and hence $q = \text{const.}$ and $\alpha - \beta = \text{const.} = \alpha_0 + 2\delta - \beta_0$),

$$U(\alpha, \alpha-\alpha_0+\beta_0-2\delta) = V(\alpha, \alpha-\alpha_0+\beta_0-2\delta)$$

* In contrast to the conventions of § 3.





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by (5), (6), (7) and (13). Together with the recurrence relations (37) and (38), these boundary conditions determine the coefficients $u_{r,s}$ and $v_{r,s}$ for region VI and the distribution of $U$ on $FG$, where $\beta = \beta_0 - 2\delta$, is found to be

$$\begin{aligned} U(\alpha, \beta_0 - 2\delta) &= -2[m_0 + 3(m_0^2 + m_1)\,\delta + (4m_0^3 + 14m_0 m_1 + 5m_2)\,\delta^2] \\ &\quad + 2(\alpha - \alpha_0)\,\{2m_0^2 + m_1 + (5m_0^3 + 13m_0 m_1 + 3m_2)\,\delta\} \\ &\quad - (\alpha - \alpha_0)^2\,(m_2 + 2m_0(m_0^2 + m_1)) + O(\delta^3) \\ &= U_2(\alpha). \end{aligned} \tag{64}$$

Region VIII is again a simple wave, and an argument analogous to the one employed for region V shows that the distribution, $U_3(\alpha)$, of $U$ on $HJ$ is

$$\begin{aligned} U_3(\alpha) &= U_2(\alpha) + m(\tfrac{1}{2}\alpha + \delta - \tfrac{1}{2}\beta_0) \\ &= -[m_0 + 2(3m_0^2 + 2m_1)\,\delta + 4(2m_0^3 + 7m_0 m_1 + 2m_2)\,\delta^2] \\ &\quad + (\alpha - \alpha_0)\,\{4m_0^2 + 3m_1 + 2(5m_0^3 + 13m_0 m_1 + 4m_2)\,\delta\} \\ &\quad - (\alpha - \alpha_0)^2\,(\tfrac{1}{2}m_2 + 2m_0(m_0^2 + m_1)) + O(\delta^3). \end{aligned} \tag{65}$$

On $HJ$, $\beta = \beta_0 - 2\delta$, so (65) prescribes $U(\alpha, \beta_0 - 2\delta)$ for region IX, the third interaction region. The other boundary condition is $U + V = 0$ on the axis, where $\alpha + \beta = 0$, as in region III, and by the help of the recurrence relations, (37) and (38), the distribution of $V$ on $JK$ is found to be

$$\begin{aligned} V(\alpha_0, \beta) &= m_0 + 4(m_0^2 + m_1)\,\delta + 8(m_2 + 3m_0 m_1 + m_0^3)\,\delta^2 \\ &\quad + (\beta - \beta_0)\,\{4m_0^2 + 3m_1 + 4(3m_0^3 + 7m_0 m_1 + 2m_2)\,\delta\} \\ &\quad + (\beta - \beta_0)^2\,(\tfrac{1}{2}m_2 + 2m_0 m_1 + m_0^3) + O(\delta^3) \\ &= V_3(\beta). \end{aligned} \tag{66}$$

Region XI, finally, is a simple wave in which $\alpha = \text{const.} = \alpha_0$. To label the plus Mach lines in this region denote by $sf(\mu_1)$ the distance from $J$ to the point where any individual plus Mach line crosses the minus Mach line $JL$ (figure 3). Then

$$\begin{aligned} V(s, \beta) &= V(\alpha_0, \beta) - sm(\tfrac{1}{2}\alpha_0 - \tfrac{1}{2}\beta) \\ &= m_0(1 - s) + 4(m_0^2 + m_1)\,\delta + 8(m_2 + 3m_0 m_1 + m_0^3)\,\delta^2 \\ &\quad + (\beta - \beta_0)\,\{(3 + s)\,m_1 + 4m_0^2 + 4(2m_2 + 7m_0 m_1 + 3m_0^3)\,\delta\} \\ &\quad + (\beta - \beta_0)^2\,(\tfrac{1}{2}(1 - s)\,m_2 + 2m_0(m_0^2 + m_1)) + O(\delta^3) \\ &= (1 - s)\,m(\tfrac{1}{2}\alpha_0 - \tfrac{1}{2}\beta) + 4\{\psi_0 + (2m_0\psi_0 + \psi_0')\,\delta\}\,(\delta + \beta - \beta_0) \\ &\quad + 2m_0\psi_0(\beta - \beta_0)\,(2\delta + \beta - \beta_0) + O(\delta^3), \end{aligned} \tag{67}$$

by (15), (66) and (34), where

$$\psi_0 = m_0^2 + m_1 = \psi(\alpha_0 - \beta_0),$$
$$\psi_0' = 2m_2 + 4m_0 m_1 = (\mathrm{d}\psi/\mathrm{d}t)_{\alpha - \beta = \alpha_0 - \beta_0},$$

by (34) and (20).

3·1·2. In inviscid, supersonic flow, the formation of shocks away from solid boundaries is associated with the presence of limit lines (Meyer 1949) and these are lines where $U = 0$ or lines where $V = 0$. Now $U$ and $V$ are both infinite in any region of uniform flow and either $U$ or $V$ is infinite in any simple wave. It is therefore to the equations (60) to (67) that we must turn in order to determine whether shock waves occur, or do not occur, in the first 'period' of a jet. It should be noted that all the series of which the first few terms are given in those equations are convergent for sufficiently small $\delta$, provided the base Mach number is neither unity nor infinite,* and the same holds for the double power series in the regions of wave interaction, for in each individual region the boundary conditions are analytic. These series are therefore also sufficient to decide the questions of shock formation and periodicity for all jets with initial Mach number, $M_N$, greater than one and pressure ratio, $p/p_N$, less than one by a sufficiently small amount.

The series (60) to (66) all start with a constant term which is non-zero and not small, by (11), and the same is true for the double power series in the regions of wave interaction. The variables $\alpha$ and $\beta$, on the other hand, vary only in an interval of magnitude $\delta$, in each region. No shock waves, therefore, can be expected in regions I to X. This does not, however, hold for the series (67) in region XI, where

$$0 \leqslant s \leqslant 1, \quad -2\delta \leqslant \beta - \beta_0 \leqslant 0. \tag{68}$$

A limit line will occur in this region if $V(s,\beta) = 0$ for any pair of values $s$, $\beta$ in (68), and the terms of first order in $\delta$ in (67) show that this condition is always satisfied if $|\psi_0| \gg \delta$. In fact, it can be shown that a shock will start from the front, $JL$, of the simple wave when $\psi_0 > 0$ and that it will start from the tail, $KM$, when $\psi_0 < 0$.

There remains a small range of initial Mach numbers, near the particular one for which $\psi = 0$, in which the question can only be decided by an examination of the terms of second order in $\delta$ in (67). This shows that the condition is again satisfied if $|\psi_0 + \delta\psi_0'| \gg \delta^2$, and that the shock will start from the front of the simple wave, if $\psi_0 + \delta\psi_0' > 0$, and from the tail if $\psi_0 + \delta\psi_0' < 0$. Thus, only an interval $O(\delta^2)$ near a certain initial Mach number, $M_N(\gamma)$, is left in which the terms given explicitly in (67) are not sufficient to decide the question of shock formation completely.

Those terms suffice, however, to decide for all Mach numbers whether the jet is periodic as predicted by Prandtl (1904) and Hasimoto (1953) for pressure ratios sufficiently near unity. A necessary condition for this periodicity is that the simple wave XI be centred at a point symmetrical to $A$ and hence

$$V(1,\beta) \equiv 0 \quad \text{for} \quad -2\delta \leqslant \beta - \beta_0 \leqslant 0. \tag{69}$$

Equation (67) shows that unless $\psi_0' = O(\delta)$, $V(1,\beta)$ cannot be uniformly small even compared with $\delta^2$. But it is not difficult to show from (20) and (11) that $d\psi/dt > 0$ at all supersonic Mach numbers for any perfect gas with $\gamma > 1$, and hence (69) cannot be satisfied if $\delta$ is sufficiently small. A closer examination of Hasimoto's (1953) theory shows, in fact, that his approximation implies neglect of terms $O(\delta^2)$ in the position co-ordinates corresponding to any given velocity. The error in $U$ and $V$ is thus $O(\delta)$, and an equivalent solution is obtained from the trivial approximation, $U =$ const. and $V =$ const., to the solution of (14) and (15) for wave-interaction regions. Such an approximation, although more accurate than linearized theory (Prandtl 1904; Pack 1950), is inadequate for an explanation of the aperiodicity of the jet and of shock formation in it.

* The limitation on $\delta$ is stringent for near-sonic and hypersonic base Mach numbers.

### 3·2. Reflexion of a simple wave from a curved streamline

To understand the application of the theory to problems with floating boundaries in the hodograph plane it may be best to consider a case containing some of the essentials of the aerofoil problem for a non-uniform incident stream. Assume that the shape of a curved streamline, $S$, is prescribed in the flow plane, i.e. $x = x_s(\theta)$ and $y = y_s(\theta)$ are given on $S$, with $\mathrm{d}y_s/\mathrm{d}x_s = \tan\theta$. Since this is not sufficient to determine a flow, assume also that a minus characteristic, $\mathfrak{M}$, is specified (figure 6*a*), i.e. $\beta =$ const. and $\alpha = \alpha_M(x)$ (or $x = x_M(\alpha)$) are prescribed on it. For convenience, let the point of intersection of $\mathfrak{M}$ and $S$ be chosen as origin in both the physical and the characteristic plane, and also as base point for the Riemann functions.

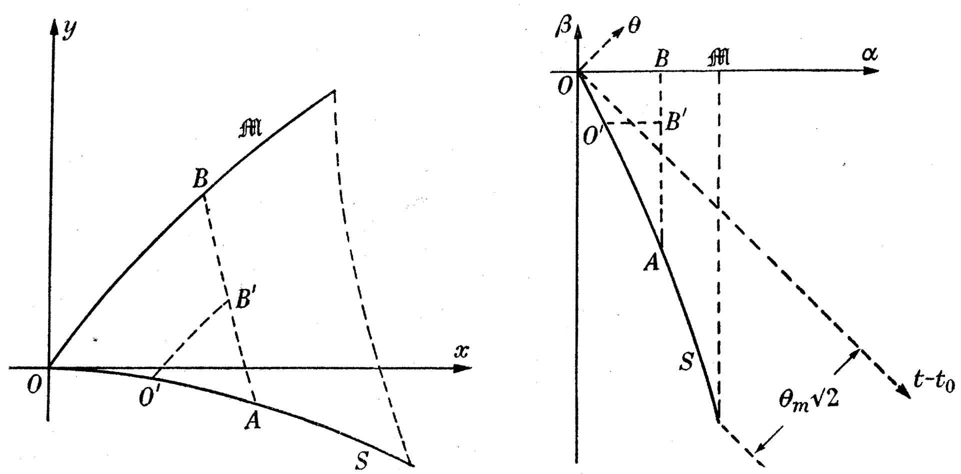


Figure 6*a*. Flow plane. Figure 6*b*. Characteristic plane.

The classical procedure is to suppose that $t = t_s(\theta)$ is also known on $S$, which is then a Cauchy boundary; to employ the solution, by Riemann's method, of this Cauchy problem to calculate $\alpha(x)$ on $\mathfrak{M}$ and so to obtain an integral equation for $t_s(\theta)$; to solve this integral equation in order to find the pressure distribution on $S$; and to substitute the result into the solution of the Cauchy problem in order to obtain the complete solution, if required.

Accordingly, we shall employ supposed knowledge of $t_s(\theta)$ to calculate $U_M(\alpha)$, the distribution of $U$ on $\mathfrak{M}$, which is known from

$$\mathrm{d}x_M/\mathrm{d}\alpha = U_M f(\mu_M)\cos(\theta_M+\mu_M) \tag{70}$$

by (6) and (13), a suffix $M$ indicating that values are to be taken at $\beta = 0$. The Cauchy data required are $U_s(\theta)$ and $V_s(\theta)$, the values of $U$ and $V$ on $S$, which are found from the equation of the streamline $S$ in the characteristic plane,

$$U_s\,\mathrm{d}\alpha_s/\mathrm{d}\theta = V_s\,\mathrm{d}\beta_s/\mathrm{d}\theta, \tag{71}$$

by (5), (6) and (13), and from

$$\mathrm{d}x_s/\mathrm{d}\theta = 2U_s f(\mu_s)\cos\theta\cos\mu_s\,\mathrm{d}\alpha_s/\mathrm{d}\theta, \tag{72}$$

by (1), (2), (5), (6), (13) and (71), where

$$\alpha_s = \theta + t_s(\theta) - t_0, \quad \beta_s = \theta - t_s(\theta) + t_0, \tag{73}$$

and a suffix $s$ indicates that values are taken on $S$. Riemann's method (Courant & Hilbert 1937), applied to (17) and (18), gives

$$Z(B) = Z(A) - \int_0^A (ZR_\xi \, d\xi + Z_\eta R \, d\eta),$$

$$Y(B) = Y(A) - \int_0^A (YS_\xi \, d\xi + Y_\eta S \, d\eta),$$

where $OAB$ is the Mach triangle based on the segment $OA$ of $S$ (figure 6$b$). Hence

$$U_M(\alpha_s) = U_s(\theta) - \int_0^\theta [m(t'_s)\, R + \tfrac{1}{2}(R_\xi + S_\xi - R_\eta + S_\eta)]\, U_s(\theta')\, (1 + dt'_s/d\theta')\, d\theta'^*$$

$$= U_s(\theta) - \int_0^\theta G(\theta'; \alpha_s, 0)\, d\theta', \tag{74}$$

by (14) to (16), (23), (24) and (71), where $t'_s = t_s(\theta')$ and the argument of the Riemann functions and their partial derivatives is $(\xi, \eta; \alpha, \beta) = (\theta' + t'_s, \theta' - t'_s; \alpha_s, 0)$. This is the (exact) integral equation for $t_s(\theta)$, and hence for the pressure distribution on the streamline.

To solve the integral equation, consider a 'thin body', i.e. one on the surface of which $\theta$ is everywhere small. Assume that $\alpha_M(x)$ is also small, then the solution is confined to a small region in the characteristic plane (figure 6$b$) and the problem is of the type considered by linearized theory. For definiteness, assume that $\alpha_M = O(\theta)$. Then if the Mach number at $O$ (figure 6$a$) is in the medium Mach number range, the Riemann functions are $1 + O(\theta^2)$ and the first approximation to (74) is

$$U_M(\alpha_s) = U_s(\theta)\,(1 + O(\theta)), \tag{75}$$

that is, the solution is obtained directly from the boundary conditions, without the Riemann functions entering into the calculations at all. If terms $O(\theta)$ are neglected against unity also in $f(\mu)$ and the trigonometric functions in (70) and (72), then (75) implies

$$\tfrac{1}{2}x_s(\theta) = x_M(\theta + t_s), \tag{76}$$

and hence a first approximation to the solution of (74) is

$$\alpha_s^{(1)}(\theta) = \alpha_M(\tfrac{1}{2}x_s(\theta)), \quad t_s^{(1)} = t_0 + \alpha_s^{(1)} - \theta. \tag{77}$$

3·2·1. It is not immediately clear whether such an approximation can be of any practical use. Since terms $O(\theta)$ have been neglected against unity in some of the factors in equations (70) and (72), it appears doubtful whether such terms can retain any meaning in the other factors. But if, for consistency, terms $O(\theta)$ are neglected against unity in all factors in these equations, then the approximation admits only streamline shapes consisting of straight-line and circular-arc segments. Thus the approximation is, in principle, of the same order as linearized theory, but, in fact, it may suffer from a lack of detail that is unacceptable.

It should be realized that this difficulty is not particular to the present theory, nor even to supersonic flow. Linearized theory does not encounter the difficulty, since it employs the boundary condition on the streamline in the form

$$\theta = \theta(x). \tag{78}$$

* If $x_s(\theta)$ or $x_M(\alpha)$ are not one-valued functions, an auxiliary parameter must be employed instead of $\theta$ or $\alpha$.

Exact theory must, as the price paid for linearization of the equations of motion, use them in the form

$$x = x(\theta). \tag{79}$$

If $\theta$ and its derivatives with respect to $x$ are small, then the derivatives of $x$ with respect to $\theta$ are large, and many terms of the expansion of $x$ in powers of $\theta$ are needed to represent $x(\theta)$ adequately. Hence if the theory is built up as a process of successive approximations, accurate to successively higher order in $\theta$, then any boundary condition of the form (79) stands in danger of losing definition of detail in the lower-order approximations.

To understand how the trap can be avoided, note that the small parameter of linearized theory is, strictly speaking, the maximum absolute value, $\theta_m$, of $\theta$ on the streamline, or some equivalent *constant*. Accordingly, put $\theta = \vartheta\theta_m$,

and expand the boundary conditions in series with respect to $\theta_m$,

$$x_s(\theta) = \sum_{j=0}^{\infty} X_j(\vartheta)\,\theta_m^j \tag{80}$$

and

$$x_M(\alpha) = \sum_{j=0}^{\infty} \Xi_j(\alpha/\theta_m)\,\theta_m^j,* \tag{81}$$

which will converge for sufficiently small $\theta_m$.† Similarly, $f(\mu)\cos(\theta_M+\mu_M)$ in (70) is a function of $\alpha$ and can be expanded in a series of the type (81) and hence $U_M$ is obtained in the form of such a series. With $\alpha_M$ assumed $O(\theta_m)$, at most, the maximum of $t_s(\theta)-t_s(0)$ must be $O(\theta_m)$ (see figure 6*b*), and hence, it is plausible to expand $t_s(\theta)$ in a series

$$t_s(\theta) = t_0 + \sum_{j=1}^{\infty} T_j(\vartheta)\,\theta_m^j \quad (t_0 = t_s(0)). \tag{82}$$

If it converges, then it follows first that

$$\alpha_s(\theta) = \sum_{j=1}^{\infty} A_j(\vartheta)\,\theta_m^j \quad \text{with} \quad A_1 = T_1+\vartheta;\ A_j = T_j \text{ for } j>1, \tag{83}$$

by (73), and secondly that $f(\mu_s)\cos\theta\cos\mu_s$ and $m(t_s)$ can be expanded in similar series, and hence that $U_s(\theta)$ is also obtained in this form. Finally, it is seen from figure 6*b* that the maximum values of the variables $\nu$, $\rho$ and $\sigma$ of § 2·1·1 are $O(\theta_m)$ and therefore $R_q^{(p)} = O(\theta_m^p)$, and the Riemann functions are also given in the form of power series with respect to $\theta_m$ by (32) and the corresponding expression for $S$.

All these series can now be substituted in the integral equation (74), and the coefficients of equal powers of $\theta_m$ compared on both sides to obtain successively the functions $T_j(\vartheta)$ by quadratures.

3·2·2. The only remaining ambiguity concerns the way in which the coefficients in the series (80) and (81) should be determined. This is a matter for choice and no generality is lost by putting

$$X_0(\vartheta) = x_s(\theta),$$

$$\Xi_0(\alpha/\theta_m) = x_M(\alpha),$$

and

$$X_j(\vartheta) \equiv \Xi_j(\alpha/\theta_m) \equiv 0 \quad \text{for} \quad j \geqslant 1.$$

* It is natural to begin the series (80) with a term independent of $\theta_m$, since the chord of the streamline segment, $S$, is usually taken as unit of length. For definiteness, it is assumed that $x_M = O(x_s)$.

† It is assumed that branch lines, if present, have been dealt with, for instance, by the method described in §2·3.

In this way, (77) is obtained in the more precise notation

$$\left.\begin{aligned} A_1(\vartheta) &= \alpha_M(\tfrac{1}{2}x_s(\theta))/\theta_m, \\ T_1(\vartheta) &= A_1-\vartheta, \end{aligned}\right\} \tag{77a}$$

which shows that (76) is to be understood to mean

$$\tfrac{1}{2}x_s(\theta) = x_M(\theta+t_s(\theta)-t_0+O(\theta_m^2)). \tag{76a}$$

Equations (76) and (77) are the result predicted by linearized theory for the wave interaction caused by the reflexion of a simple wave from a curved streamline (figure 7).* It is seen, therefore, that the procedure proposed in the preceding section for the solution of the integral equation (74) does not introduce any lack of detail.

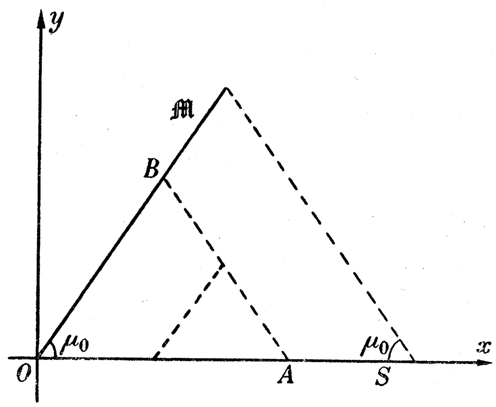


FIGURE 7. Flow plane of linearized theory.

The second approximation is obtained by neglecting terms $O(\theta_m^2)$ against unity. It is still sufficient to put $R \equiv S \equiv 1$ in (74), and the result is

$$\alpha_s^{(2)}(\theta) = A_1(\vartheta)\,\theta_m + A_2(\vartheta)\,\theta_m^2 = \alpha_M(\tfrac{1}{2}k(\theta)\,x_s(\theta)),$$

$$t_s^{(2)}(\theta) = t_0 + T_1(\vartheta)\,\theta_m + T_2(\vartheta)\,\theta_m^2 = \alpha_s^{(2)}(\theta) - \theta + t_0,$$

with $$k(\theta) = 1 + (2m_0 - \tan\mu_0)\left[\theta - \frac{1}{x_s}\int_0^\theta x_s(\theta')\,d\theta'\right] - m_0\,\alpha_M(\tfrac{1}{2}x_s(\theta)),$$

where a suffix 0 denotes values taken at $O$ (figure 6).

Higher approximations can be similarly obtained. The convergence of the series (82) can be examined by Picard's method, since the procedure for solving (74) is closely related to an iteration method.

It is now seen that the expansions of the preceding section are not required in practice. What is required is that the terms in equations (70), (72) and (74) be divided in two classes. The first class consists of the Riemann functions, $m(t)$, $f(\mu)$, and the trigonometric functions (which stem from (5) and (6)); all these may be said to represent the contribution to our formulae of the differential equations. The second class consists of $x_M$, $x_s$, $U_M$, $U_s$, $V_s$, $\alpha_s$, $\beta_s$

* Ackeret's theory concerns the special case where the incident wave can be approximated by uniform flow, and the difference between Busemann's and Ackeret's theories lies in the approximation used for the relation between $t$ and the pressure.

and $t_s$, and these functions represent, or depend on, the boundary conditions. In order to obtain a scheme of successive approximations without loss of detail, it is sufficient to adopt the 'inconsistent' rule that all functions of the first class should be approximated by polynomials of successively higher degree in the characteristic variables, or $\theta$, or $t$, but that the functions of the second class should *not* be approximated.

2·2·3. It remains to clarify the relation of the present theory to Whitham's (1952) uniform first-order theory of a simple wave. As far as the pressure on the streamline, $S$, is concerned, Whitham's theory is identical with Ackeret's (1925). An equation equivalent to Whitham's for the shape of the minus Mach lines, $\beta = \text{const.} = \beta_s$, is obtained from (2),

$$\mathrm{d}x/\mathrm{d}y = \cot(\theta+\mu).$$

If terms $O(\theta_m)$ are neglected,

$$\mathrm{d}x/\mathrm{d}y = \cot\mu_0 \quad \text{and} \quad x - y\cot\mu_0 = \text{const.} = x_s - y_s\cot\mu_0,$$

where a suffix $s$ indicates values taken on the streamline and a suffix 0 values taken at $O$ (figure 6*a*). This is the approximation of linearized theory for these Mach lines. If terms $O(\theta_m^2)$ only are neglected,

$$\mathrm{d}x/\mathrm{d}y = \cot\mu_0[1 - 2m_0\beta_s + 2(m_0 - \operatorname{cosec} 2\mu_0)\,\alpha], \tag{84}$$

by (1), (2) and (11), and

$$x - y\cot\mu_0 = x_s - y_s\cot\mu_0 - 2(y-y_s)\,\beta_s m_0\cot\mu_0 + 2\cot\mu_0(m_0 - \operatorname{cosec} 2\mu_0)\int_{\alpha_s}^{\alpha}\alpha'\frac{\partial y}{\partial\alpha'}\,\mathrm{d}\alpha'. \tag{85}$$

Except for the last term, (85) is identical with Whitham's equation for the minus Mach lines; his function $F(y)$ equals $\frac{1}{2}\beta_s$, in our notation, and his characteristic parameter $y$ equals $x_s - y_s\cot\mu_0$. The last term is due to the incident simple wave, which introduces a first-order curvature of the Mach lines. For the problem considered by Whitham, $\alpha \equiv 0$ and $\beta_s = 2\theta$, and (85) suffices for the prediction of shocks. For the general problem of wave interaction, however, the solution of the integral equation, (74), is required to furnish the relation between $\beta_s(\theta) = 2\theta - \alpha_s(\theta)$* and the co-ordinates $x_s(\theta)$, $y_s(\theta)$, of the streamline, $S$. Moreover, an additional relation must be derived from the full solution of the wave-interaction problem to furnish a relation between $\alpha$ and $y$ (or $x$) on the minus Mach line $\beta = \beta_s$.

While the prediction of shock shape is deferred to part II below, it may be instructive to describe briefly the prediction of shock occurrence, as an example of a problem requiring knowledge of the full solution, not only of the pressure distribution on the streamline, $S$. For this purpose, it is more convenient not to proceed from (85), but to define the limit lines directly as lines of infinite velocity gradient, that is, lines $U = 0$ or $V = 0$.

The solution for $U$ is obtained by applying Riemann's method to the general point $B'$ (figure 6*a*), rather than the point $B$, which gives

$$U(\alpha,\beta) = U_s(\theta_s(\alpha)) - \int_{\theta_s(\beta)}^{\theta_s(\alpha)} G(\theta'; \alpha,\beta)\,\mathrm{d}\theta', \tag{86}$$

* For consistency with the accuracy of (85), the first-order approximation giving the term $O(\theta_m)$ in the pressure distribution must be used, that is, again the pressure distribution of Ackeret's theory.

instead of (74). Here $\theta_s(\alpha)$ and $\theta_s(\beta)$ denote respectively the values of $\theta$ on the streamline, $S$, at the points $A$ and $O'$, where $\alpha$ and $\beta$ take the same values as at $B'$ (figure 6 $a$, $b$). Subtraction of (74) from (86) gives

$$U(\alpha,\beta) = U_M(\alpha) + \int_0^{\theta_s(\alpha)} G(\theta';\alpha,0)\,\mathrm{d}\theta' - \int_{\theta_s(\beta)}^{\theta_s(\alpha)} G(\theta';\alpha,\beta)\,\mathrm{d}\theta'$$

$$= U_M(\alpha)\,(1+O(\theta_m)), \tag{87}$$

by (75) and (77), and since $\alpha_M = O(\theta_m)$. The next approximation for $U(\alpha,\beta)$ is obtained by neglecting only terms $O(\theta_m)$ in $G$, which yields, by (74) and (72),

$$U(\alpha,\beta) = U_M(\alpha) + \int_0^{\theta_s(\beta)} \frac{m_0(1+O(\theta_m))}{2f(\mu_0)\cos\mu_0}\frac{\mathrm{d}x_s}{\mathrm{d}\theta'}\,\mathrm{d}\theta',$$

and by (70) $$f(\mu_0)\cos\mu_0 U(\alpha,\beta) = \{\mathrm{d}x_M/\mathrm{d}\alpha + \tfrac{1}{2}m_0 x_s(\theta_s(\beta))\}\,(1+O(\theta_m)). \tag{88}$$

An analogous expression for $V$ is best obtained directly by integrating (15) from $O'$ to $B'$ (figure 6 $a$, $b$), which gives

$$V(\alpha,\beta) = V_s(\beta) - \int_{\alpha_s(\beta)}^{\alpha} m(\tfrac{1}{2}\alpha' - \tfrac{1}{2}\beta)\,U(\alpha',\beta)\,\mathrm{d}\alpha'$$

$$= V_s(\beta) - m_0\int_{\alpha_s(\beta)}^{\alpha} U_M(\alpha')\,(1+O(\theta_m))\,\mathrm{d}\alpha',$$

by (87). Here $V_s(\beta)$ and $\alpha_s(\beta)$ denote respectively the values of $V$ and $\alpha$ at the point $O'$ of the streamline. Now, $V_s(\beta)$ can be expressed in terms of $\mathrm{d}x_s/\mathrm{d}\theta$, by (71) and (72), and the integral can be evaluated by the help of (70), whence

$$f(\mu_0)\cos\mu_0\,V(\alpha,\beta) = \left\{\frac{\mathrm{d}x_s/\mathrm{d}\theta}{2\,\mathrm{d}\beta_s/\mathrm{d}\theta} - m_0[x_M(\alpha) - x_M(\alpha_s(\beta))]\right\}(1+O(\theta_m)). \tag{89}$$

By (89), lines $V=0$, i.e. limit lines enveloping the family of minus Mach lines, will occur when

$$\mathrm{d}x_s/\mathrm{d}\beta_s > 0, \tag{90}$$

and the co-ordinates, $x_l(\beta)$ and $y_l(\beta)$, of the limit point on any Mach line $\beta=$ const. will be given by

$$\left.\begin{aligned} x_l(\beta) - x_s(\theta_s(\beta)) &= \frac{1}{2m_0}\frac{\mathrm{d}x_s}{\mathrm{d}\beta_s}(1+O(\theta_m)),\\ x_l(\beta) - y_l(\beta)\cot\mu_0 &= (x_s(\beta) - y_s(\beta)\cot\mu_0)\,(1+O(\theta_m)),\end{aligned}\right\} \tag{91}$$

provided the limit point lies within the region in which the boundary conditions determine the solution. It will be noted that (91) is identical with the prediction obtained directly for the envelope of the Mach line family (85) by neglecting the last term on the right-hand side of that equation, and hence the curvature of the Mach lines has no influence on the process of limit-line formation. But the prediction is not identical with that obtained for a simple wave (Whitham 1952), where $\beta_s = 2\theta$ and (90) means that the streamline, $S$, is concave with respect to the flow. In the general case, concavity is neither necessary, nor sufficient, due to the influence which the incident wave can have on the process of limit-line formation.

Moreover, shock waves may occur in the general case which strike the streamline, $S$, and are reflected from it. They are due entirely to the structure of the incident wave, but may start in the region of wave interaction. Their occurrence is indicated by limit lines $U=0$ and, by (88), these occur in the region of interaction if

$$\mathrm{d}x_M/\mathrm{d}\alpha = -\tfrac{1}{2}m_0 x_s(\theta_s(\beta))$$

for a point $B'$ between $\mathfrak{M}$ and $S$ (figure 6*a*). This means, first,

$$\mathrm{d}x_M/\mathrm{d}\alpha < 0,$$

so that $O'$ lies to the right of $O$, i.e. a part, at least, of the incident wave must be a wave of compression, and secondly,

$$\mathrm{d}x_M/\mathrm{d}\alpha > -m_0 x_M(\alpha),$$

by (76*a*), so that $O'$ lies to the left of $A$.

It would appear at first sight that, since the assumptions $\theta = O(\theta_m)$, $\alpha_M = O(\theta_m)$ and $x_M = O(x_s)$ imply that both $\mathrm{d}x_s/\mathrm{d}\beta_s$ and $\mathrm{d}x_M/\mathrm{d}\alpha$ are $O(\theta_m^{-1})$, there can be no limit points within the finite region of interaction. The assumptions do not, however, exclude the possibility that $\theta_m \mathrm{d}x_s/\mathrm{d}\beta_s$ and $\theta_m \mathrm{d}x_M/\mathrm{d}\alpha$ are $\ll 1$ for *some* values of $\theta$ and $\alpha$, respectively. The results obtained furnish, therefore, an extension to the wave interaction, with proof, of the result regarding prediction of shock occurrence which Whitham derived from his hypothesis. The proof shows also that the result is independent of the assumption—implicit in linearized theory, but not implied by the thin-body concept—that the velocity gradients are small. In particular, the pressure distribution predicted by linearized theory on the streamline, $S$, is correct to $O(\theta_m)$, independently of that assumption. In fact, since zeros* of $\theta_m \mathrm{d}x_s/\mathrm{d}\theta$ are admitted, the results extend to the case of streamlines with (isolated) discontinuities of slope.

It may be worth noting, moreover, that the assumptions that $\alpha_M = O(\theta_m)$ and $x_M = O(x_s)$ are not necessary. When they do not apply, the power series of § 3·2·1 must be modified, but the results given (except for the error terms) extend both to Whitham's case where the incident wave is weak† and the reflected wave, therefore, nearly a simple wave, and to the case of a simple wave emerging from the interaction. In both cases, the boundary conditions do not restrict the solution to a finite region.

### 3·3. Treatment by double power series

Understanding of the method of double power series (see § 2·2) will also be improved by considering briefly its application to a problem with a floating boundary condition in the characteristic plane. In this method, both the solution $U$, $V$ and the boundary conditions are expanded in power series with respect to $\alpha$ and $\beta$. To avoid loss of detail, a relatively large number of terms must be retained in all the series representing boundary conditions, even when very few terms in the series (34), (35) and (36) would suffice for an adequate representation of the differential equations. For lower-order approximations, therefore, Riemann's method appears preferable. As the desired order of accuracy increases, however, the double power series become more and more attractive. For, on the one hand, the integral equation thrown up by Riemann's method becomes more cumbersome to solve, and, on the other hand, the danger of loss of detail recedes.

* They need not be isolated.

† In that case, the error term in (88) is such as to invalidate the conclusions drawn with regard to the formation of limit lines $U=0$.

To apply the method of double power series to the reflexion of a simple wave from a curved streamline of given shape the following procedure may be adopted. By (35)

$$U_M(\alpha) = U(\alpha, 0) = \sum_{r=0}^{\infty} u_{r,0}\,\alpha^r,$$

since $\alpha_0 = \beta_0 = 0$ in (34) to (36), if the common point of $\mathfrak{M}$ and $S$ (figure 6 *a*, *b*) is again chosen as base point and origin in the characteristic plane, and the coefficients $u_{r,0}$ can be found for all $r$ from (70). In (72) the functions $\sec\theta\,dx_s/d\theta$ and $f(\mu_s)\cos\mu_s$ can be expanded in series (with known coefficients) in $\theta$ and $t_s$, respectively. It has been shown in §§ 3·2·1 and 3·2·2 that $t_s(\theta)$ possesses an expansion, (82), which may more briefly be written

$$t_s(\theta) = \sum_{k=0}^{\infty} t_k\,\theta^k.$$

By its help, $f(\mu_s)\cos\mu_s$ may now be expanded directly in a series with respect to $\theta$, and, by (73), (35) and (36), similar series with respect to $\theta$ may be set up for $\alpha_s$, $\beta_s$, $U_s$ and $V_s$. All these series may be substituted into (71) and (72), terms may be rearranged, and the coefficients of each power of $\theta$ may be compared separately.

The resulting relations, together with (37) and (38), form a system of equations which is sufficient to determine the unknown coefficients, $u_{r,s}$, $v_{r,s}$ and $t_k$ for all $r$, $s$ and $k$, and which has the following two properties: (i) The complete set of equations separates into subsets, one for each integer $n$, each of which involves the $u_{r,s}$, $v_{r,s}$ and $t_k$ with $r+s \leqslant n$ and $k \leqslant n$, but none with $r+s > n$ or $k > n$. (ii) Each subset is a system of equations linear in $t_n$ and the $u_{r,s}$ and $v_{r,s}$ with $r+s = n$. That is, the unknown coefficients can be computed successively for each order of approximation, and, each time, a linear system of equations has to be solved. The double power series method therefore presents the numerical problem in a form particularly suitable for treatment by means of high-speed computing machines with established routines for the inversion of matrices.

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## Appendix

The following tables give the Mach number, $M$, the Mach angle, $\mu$, and the functions $f(\mu)$ (equation (12)), $\phi(\alpha-\beta)$ (equation (19)), and $\psi(\alpha-\beta)$ (equation (20)) in terms of $t=\frac{1}{2}(\alpha-\beta)$ (equation (3)), for $\gamma = 1{\cdot}3$ (table 2) and $\gamma = 1{\cdot}4$ (table 3). More detailed tables of $\mu$ in terms of $1000°-t$, for $\gamma = 1{\cdot}4$, are given by Herbert & Older (1946).

Table 2 ($\gamma=1{\cdot}3$)

| $t$ | $M$ | $\mu$ (deg.) | $f(\mu)$ | $\phi$ | $\psi$ |
|---|---|---|---|---|---|
| 0 | 1·0000 | 90 | ∞ | ∞ | −∞ |
| 0·05 | 1·1651 | 59·1219 | 8·6756 | 24·573 | −12·290 |
| 0·10 | 1·2725 | 51·7991 | 8·1071 | 7·0886 | −2·3032 |
| 0·15 | 1·3692 | 46·9147 | 7·9418 | 3·5804 | −0·5297 |
| 0·20 | 1·4612 | 43·1860 | 7·9415 | 2·2777 | +0·0713 |
| 0·25 | 1·5509 | 40·1508 | 8·0385 | 1·6474 | 0·3465 |
| 0·30 | 1·6396 | 37·5834 | 8·2064 | 1·2946 | 0·4997 |
| 0·35 | 1·7282 | 35·3553 | 8·4339 | 1·0788 | 0·5988 |
| 0·40 | 1·8173 | 33·3855 | 8·7159 | 0·93943 | 0·6715 |
| 0·45 | 1·9074 | 31·6192 | 9·0510 | 0·84663 | 0·7307 |
| 0·50 | 1·9989 | 30·0176 | 9·4404 | 0·78429 | 0·7831 |
| 0·55 | 2·0922 | 28·5521 | 9·8871 | 0·74305 | 0·8325 |
| 0·60 | 2·1876 | 27·2009 | 10·395 | 0·71723 | 0·8810 |
| 0·65 | 2·2855 | 25·9472 | 10·971 | 0·70317 | 0·9304 |
| 0·70 | 2·3861 | 24·7774 | 11·622 | 0·69855 | 0·9816 |
| 0·75 | 2·4898 | 23·6806 | 12·356 | 0·70175 | 1·0354 |
| 0·80 | 2·5970 | 22·6479 | 13·186 | 0·71177 | 1·0926 |
| 0·85 | 2·7079 | 21·6718 | 14·123 | 0·72796 | 1·1539 |
| 0·90 | 2·8231 | 20·7461 | 15·184 | 0·74989 | 1·2200 |
| 0·95 | 2·9428 | 19·8653 | 16·388 | 0·77737 | 1·2913 |
| 1·00 | 3·0677 | 19·0250 | 17·756 | 0·81039 | 1·3688 |
| 1·05 | 3·1981 | 18·2213 | 19·317 | 0·84907 | 1·4531 |
| 1·10 | 3·3347 | 17·4505 | 21·103 | 0·89362 | 1·5452 |
| 1·15 | 3·4780 | 16·7098 | 23·154 | 0·94445 | 1·6460 |
| 1·20 | 3·6287 | 15·9965 | 25·519 | 1·0020 | 1·7568 |
| 1·25 | 3·7878 | 15·3081 | 28·260 | 1·0670 | 1·8788 |
| 1·30 | 3·9559 | 14·6427 | 31·450 | 1·1402 | 2·0135 |
| 1·35 | 4·1341 | 13·9983 | 35·184 | 1·2226 | 2·1629 |
| 1·40 | 4·3235 | 13·3732 | 39·578 | 1·3160 | 2·3290 |
| 1·45 | 4·5255 | 12·7659 | 44·781 | 1·4198 | 2·5145 |
| 1·50 | 4·7416 | 12·1750 | 50·985 | 1·5379 | 2·7224 |
| 1·55 | 4·9735 | 11·5994 | 58·432 | 1·6716 | 2·9564 |
| 1·60 | 5·2232 | 11·0377 | 67·446 | 1·8236 | 3·2210 |
| 1·65 | 5·4932 | 10·4889 | 78·446 | 1·9971 | 3·5219 |
| 1·70 | 5·7862 | 9·9521 | 91·997 | 2·1961 | 3·8658 |
| 1·75 | 6·1058 | 9·4263 | 108·86 | 2·4256 | 4·2612 |
| 1·80 | 6·4559 | 8·9108 | 130·08 | 2·6918 | 4·7190 |
| 1·85 | 6·8417 | 8·4046 | 157·10 | 3·0028 | 5·2531 |
| 1·90 | 7·2690 | 7·9073 | 192·00 | 3·3692 | 5·8811 |
| 1·95 | 7·7456 | 7·4179 | 237·75 | 3·8046 | 6·6264 |
| 2·00 | 8·2810 | 6·9359 | 298·75 | 4·3270 | 7·5201 |
| 2·05 | 8·8871 | 6·4607 | 381·64 | 4·9610 | 8·6043 |
| 2·10 | 9·5798 | 5·9918 | 496·75 | 5·7412 | 9·9370 |
| 2·15 | 10·3797 | 5·5285 | 660·67 | 6·7153 | 11·601 |
| 2·20 | 11·3147 | 5·0705 | 900·90 | 7·9532 | 13·714 |

Table 3 ($\gamma = 1{\cdot}4$)

| $t$ | $M$ | $\mu$ (deg.) | $f(\mu)$ | $\phi$ | $\psi$ |
|---|---|---|---|---|---|
| 0 | 1·0000 | 90 | $\infty$ | $\infty$ | $-\infty$ |
| 0·05 | 1·1711 | 58·63632 | 6·5847 | 24·6386 | −12·2189 |
| 0·10 | 1·2838 | 51·16376 | 6·1718 | 7·11899 | −2·2408 |
| 0·15 | 1·3863 | 46·16408 | 6·0650 | 3·60132 | −0·46709 |
| 0·20 | 1·4848 | 42·33572 | 6·0844 | 2·29484 | +0·13747 |
| 0·25 | 1·5819 | 39·20992 | 6·1803 | 1·66321 | 0·41820 |
| 0·30 | 1·6789 | 36·55817 | 6·3335 | 1·31043 | 0·57840 |
| 0·35 | 1·7768 | 34·24988 | 6·5362 | 1·09565 | 0·68597 |
| 0·40 | 1·8765 | 32·20283 | 6·7854 | 0·95807 | 0·76863 |
| 0·45 | 1·9784 | 30·36151 | 7·0814 | 0·86782 | 0·83939 |
| 0·50 | 2·0833 | 28·68647 | 7·4263 | 0·80880 | 0·90512 |
| 0·55 | 2·1915 | 27·14866 | 7·8243 | 0·77175 | 0·96981 |
| 0·60 | 2·3038 | 25·72609 | 8·2806 | 0·75107 | 1·03605 |
| 0·65 | 2·4206 | 24·40147 | 8·8025 | 0·74326 | 1·10571 |
| 0·70 | 2·5425 | 23·16115 | 9·3988 | 0·74615 | 1·18027 |
| 0·75 | 2·6702 | 21·99395 | 10·081 | 0·75838 | 1·26111 |
| 0·80 | 2·8044 | 20·89079 | 10·861 | 0·77920 | 1·34956 |
| 0·85 | 2·9458 | 19·84409 | 11·758 | 0·80828 | 1·44703 |
| 0·90 | 3·0955 | 18·84743 | 12·791 | 0·84563 | 1·55506 |
| 0·95 | 3·2544 | 17·89539 | 13·985 | 0·89155 | 1·67544 |
| 1·00 | 3·4236 | 16·98320 | 15·373 | 0·94666 | 1·81026 |
| 1·05 | 3·6045 | 16·10690 | 16·994 | 1·01185 | 1·96197 |
| 1·10 | 3·7987 | 15·26285 | 18·898 | 1·08833 | 2·13358 |
| 1·15 | 4·0080 | 14·44802 | 21·150 | 1·17772 | 2·32871 |
| 1·20 | 4·2345 | 13·65968 | 23·833 | 1·28209 | 2·55185 |
| 1·25 | 4·4809 | 12·89525 | 27·055 | 1·40411 | 2·80863 |
| 1·30 | 4·7502 | 12·15268 | 30·959 | 1·54718 | 3·10607 |
| 1·35 | 5·0462 | 11·42991 | 35·736 | 1·71571 | 3·45320 |
| 1·40 | 5·3735 | 10·72512 | 41·647 | 1·91546 | 3·86168 |
| 1·45 | 5·7379 | 10·03680 | 49·053 | 2·15398 | 4·34672 |
| 1·50 | 6·1465 | 9·36336 | 58·460 | 2·44141 | 4·92872 |
| 1·55 | 6·6085 | 8·70350 | 70·600 | 2·79143 | 5·63516 |
| 1·60 | 7·1358 | 8·05592 | 86·550 | 3·22310 | 6·50420 |
| 1·65 | 7·7439 | 7·41949 | 107·94 | 3·7633 | 7·5897 |
| 1·70 | 8·4542 | 6·79305 | 137·33 | 4·4511 | 8·9698 |
| 1·75 | 9·2957 | 6·17565 | 178·84 | 5·3449 | 10·7615 |
| 1·80 | 10·309 | 5·56636 | 239·48 | 6·5355 | 13·1461 |
| 1·85 | 11·556 | 4·96418 | 331·75 | 8·1695 | 16·4174 |
| 1·90 | 13·129 | 4·36828 | 479·37 | 10·4974 | 21·0758 |
| 1·95 | 15·177 | 3·77789 | 731·07 | 13·9745 | 28·0323 |
| 2·00 | 17·959 | 3·19209 | 1198·0 | 19·5031 | 39·0914 |
| 2·05 | 21·958 | 2·61029 | 2169·8 | 29·0794 | 58·2456 |
| 2·10 | 28·207 | 2·03167 | 4566·5 | 47·8900 | 95·8680 |
| 2·15 | 39·369 | 1·45551 | 12348 | 93·1454 | 186·379 |
| 2·20 | 65·031 | 0·88109 | 55448 | 253·892 | 507·874 |
| 2·25 | 186·19 | 0·30773 | 129900 | 2080·13 | 4160·36 |



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